Landau's Theorem revisited

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Abstract

Two new elementary proofs are given of Landau's Theorem on necessary and sufficient conditions for a sequence of integers to be the score sequence for some tournament. The first is related to existing proofs by majorization, but it avoids depending on any facts about majorization. The second is natural and direct. Both proofs are constructive, so they each provide an algorithm for obtaining a tournament realizing a sequence satisfying Landau's conditions.

I. Introduction.

In 1953 H. G. Landau [3] proved that some rather obvious necessary conditions for a non-decreasing sequence of n integers to be the score sequence for some n-tournament are, in fact, also sufficient. Namely, the sequence is a score sequence if and only if, for each k, $1 \leq k \leq n$, the sum of the first k terms is at least $\binom{k}{2}$, with equality when k = n. There are now several proofs of this fundamental result in tournament theory, ranging from clever arguments involving gymnastics with subscripts, arguments involving arc reorientations of properly chosen arcs, arguments by contradiction, arguments involving the idea of majorization, to a constructive argument utilizing network flows and another one involving systems of distinct representatives. Two of the more well-known proofs are discussed briefly below. Many of these existing proofs are discussed in a 1996 survey by Reid [6]. The notation and terminology here will be as in that survey, except that for vertices x and y, $x \rightarrow y$ will be used to denote both an arc from x to y and the fact that x dominates y, where the context makes clear which use is intended.

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In this paper we give an elementary, self-contained proof that is related to known proofs by majorization (Aigner [1] in 1984 and Li [4] in 1986), but it does not depend on any appeals to chains and covers in lattices. And, we give a new, direct proof that employs a simple operation on sequences that is as basic as any in the literature, and perhaps more natural. Neither proof is in [6]. Both proofs are constructive, so they each provide an algorithm for obtaining a tournament realizing a sequence satisfying Landau's conditions.

First we give the statement of Landau's Theorem.

Theorem (Landau [3]). A sequence of integers $\mathbf{s} = (s_1 \leq s_2 \leq \ldots \leq s_n)$, $n \geq 1$, is a score sequence if and only if

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \quad 1 \leq k \leq n, \quad \text{with equality for } k = n. \tag{1}$$

All of the published proofs concern the sufficiency of conditions (1) since the necessity of those conditions follows easily from the observation that if $\mathbf{s}$ is a score sequence of some n-tournament T, then any k vertices of T form a subtournament W and, hence, the sum of the scores in T of these k vertices must be at least the sum of their scores in W which is just the total number of arcs in W, $\binom{k}{2}$.

We sketch Bang and Sharp's [2] elegant proof of the sufficiency of conditions (1). Let s be a sequence of integers satisfying conditions (1). Let $X_1, X_2, \ldots, X_n$ be n pairwise disjoint sets, where $|X_i| = s_i$, $1 \leq i \leq n$. Form the family of $\binom{n}{2}$ distinct sets $F = \{X_i \cup X_j \mid 1 \leq i < j \leq n\}$. Utilize conditions (1) and P. Hall's Theorem on systems of distinct representatives to verify that the family F of sets has a system of distinct representatives $\{a_{ij} \mid 1 \leq i < j \leq n\}$, where $a_{ij} \in X_i \cup X_j$, for all i and j, $1 \leq i < j \leq n$. Check that the n-tournament with vertex set $\{X_1, X_2, \ldots, X_n\}$ in which X_i dominates X_j if and only if $a_{ij} \in X_i$ has score sequence $\mathbf{s}$, as desired. Note that this construction rests on the construction of a certain matching in a certain bipartite graph; there are efficient algorithms for that.

We also sketch a self contained, clever proof due to Thomassen [7]. It proceeds by contradiction. Let n be the smallest integer for which there is a non-decreasing sequence $\mathbf{s}$ of non-negative integers satisfying Landau's conditions (1), but for which there is no n-tournament with score sequence $\mathbf{s}$. Among all such $\mathbf{s}$, pick one for which s_1 is as small as possible. If $\sum_{i=1}^{k} s_i = \binom{k}{2}$, for some k, $1 \leq k \leq n-1$, then check that sequences $\mathbf{s}' = (s_1, s_2, \ldots, s_k)$ and $\mathbf{s}'' = (s_{k+1}-k, s_{k+2}-k, \ldots, s_n-k)$ satisfy conditions (1) and are shorter than $\mathbf{s}$. So, there is a k-tournament V with score sequence $\mathbf{s}'$ and an (n-k)-tournament U with score sequence $\mathbf{s}''$. The n-tournament consisting of disjoint copies of U and V such that every vertex of U dominates every vertex in V has score sequence $\mathbf{s}$. On the other hand, if the inequalities in (1) are strict for all k , $1 \leq k \leq n-1$, then $\mathbf{s}''' = (s_1-1, s_2, \ldots, s_n+1)$ satisfies Landau's conditions (1). So, by choice of s_1, there is an n-tournament W with score sequence $\mathbf{s}'''$. Check that W contains a path P of length 2 from the vertex of score s_n+1 to the vertex of score s_1-1. Reversal of the 2 arcs in P results in an n-tournament with score sequence **s**, a contradiction. This proof also is the basis for an algorithm. We remark that a slightly earlier proof due to Mahmoodian [5] proceeds exactly as in Thomassen's proof up to the appearance of **s**‴.

It uses the fact that if $m=\min\{\sum_{i=1}^{k} s_i-\binom{k}{2} \mid 1\le k\le n-1\}$, then $(s_1-m, s_2, \ldots, s_n+m)$ is a score sequence. Reversal of m 2-paths from the vertex of score s_1-m to the vertex of score s_n+m yields an n-tournament with score sequence **s**, a contradiction.

II. A Majorization Proof.

Let **s** be an integer sequence satisfying conditions (1). Starting with the transitive n-tournament, denoted TT_n, we successively reverse the orientation of the two arcs in selected 2-paths until we construct a tournament with score sequence **s**.

Suppose that at some stage we have obtained n-tournament U with score sequence $\mathbf{u} = (u_1, u_2, \ldots, u_{n-1}, u_n)$, such that, for $1 \le k \le n$, $\sum_{i=1}^{k} s_i \ge \sum_{i=1}^{k} u_i$ (with equality for $k = n$). This holds initially, when $U = TT_n$, by our hypothesis concerning **s**, since TT_n has score sequence $\mathbf{t}_n = (0, 1, 2, \ldots, n-1)$. If $\mathbf{u} = \mathbf{s}$, we are done (**s** is the score sequence of U), so suppose that $\mathbf{u} \ne \mathbf{s}$. Let α denote the smallest index such that $u_\alpha < s_\alpha$. Let β denote the largest index such that $u_\beta = u_\alpha$. Since $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} u_i$ $\left(= \binom{n}{2}\right)$, by (1) there exists a smallest index $\gamma > \beta$ such that $u_\gamma > s_\gamma$. By maximality of β, $u_{\beta+1} > u_\beta$, and by minimality of γ, $u_\gamma > u_{\gamma-1}$. We have $(u_1, \ldots u_{\alpha-1}) = (s_1, \ldots, s_{\alpha-1})$, $u_\alpha = \ldots = u_\beta < s_\alpha \le \ldots \le s_\beta \le s_{\beta+1}$, $s_{\beta+1} \ge u_{\beta+1}, \ldots, s_{\gamma-1} \ge u_{\gamma-1}$, $s_\gamma < u_\gamma$, and, of course, $u_\gamma \le \ldots \le u_n$ and $s_\gamma \le \ldots \le s_n$. Then $u_\gamma > s_\gamma \ge s_\beta > u_\beta$, or $u_\gamma \ge u_\beta+2$. So, if vertex v_i in U has score u_i, $1 \le i \le n$, there must be a vertex v_λ, $\lambda \ne \beta, \gamma$, such that $v_\gamma \rightarrow v_\lambda \rightarrow v_\beta$ in U. Reversing this 2-path yields an n-tournament U′ with score sequence $\mathbf{u}' = (u_1', u_2', \ldots, u_n')$, where

$$u_i' = \begin{cases} u_\gamma - 1, & \text{if } i = \gamma; \\ u_\beta + 1, & \text{if } i = \beta; \\ u_i, & \text{otherwise.} \end{cases}$$

By choice of indices, $u_1' \le u_2' \le \ldots \le u_n'$. It is easy to check that for $1 \le k \le n$, $\sum_{i=1}^{k} s_i \ge \sum_{i=1}^{k} u_i'$.

For n-tuples of real numbers **a** and **b** recall the "Manhattan" metric $d(\mathbf{a},\mathbf{b}) = \sum_{i=1}^{n} |a_i - b_i|$. Then, for the sequences **s**, **u**, **u**′ above, $d(\mathbf{u}', \mathbf{s}) = d(\mathbf{u},\mathbf{s}) - 2$. Now, modulo 2, $d(\mathbf{u},\mathbf{s}) \equiv \sum_{i=1}^{n}(u_i - s_i) = \sum_{i=1}^{n} u_i - \sum_{i=}^{n} s_i = 0$. So, eventually, after $(1/2)d(\mathbf{t}_n,\mathbf{s})$ such steps, we arrive at $\mathbf{u} = \mathbf{s}$ and U realizes **s**. ■

III. A New Basic Proof.

The specific sequence $\mathbf{t}_n = (0, 1, 2, \ldots, n-1)$ satisfies conditions (1) as it is the score sequence of the transitive n-tournament. If sequence $\mathbf{s} \neq \mathbf{t}_n$ satisfies (1), then $s_1 \geq 0$ and $s_n \leq n-1$, so $\mathbf{s}$ must contain a repeated term. The object of this proof is to produce a new sequence $\mathbf{s}'$ from $\mathbf{s}$ which also satisfies (1), is "closer" to $\mathbf{t}_n$ than is $\mathbf{s}$, and is a score sequence if and only if $\mathbf{s}$ is a score sequence. We find the first repeated term of $\mathbf{s}$, reduce its first occurrence in $\mathbf{s}$ by 1 and increase its last occurrence in $\mathbf{s}$ by 1 in order to form $\mathbf{s}'$. The process is repeated until the sequence $\mathbf{t}_n$ is obtained. We now prove the validity of this procedure.

Let $\mathbf{s} \neq \mathbf{t}_n$ be a sequence satisfying (1). Define *k* to be the smallest index for which $s_k = s_{k+1}$, and define *m* to be the number of occurrences of the term s_k in $\mathbf{s}$. Note that $k \geq 1$ and $m \geq 2$, and that either $k + m - 1 = n$ or $s_k = s_{k+1} = \ldots = s_{k+m-1} < s_{k+m}$. Define $\mathbf{s}'$ as follows: for $1 \leq i \leq n$,

$$s_i' = \begin{cases} s_i - 1, & \text{if } i = k; \\ s_i + 1, & \text{if } i = k + m - 1; \\ s_i, & \text{otherwise.} \end{cases}$$

Then $s_1' \leq s_2' \leq \ldots \leq s_n'$.

If $\mathbf{s}'$ is the score sequence of some n-tournament T in which vertex v_i has score s_i', $1 \leq i \leq n$, then, since $s_{k+m-1}' > s_k'+1$, there is a vertex in T, say v_p, for which $v_{k+m-1} \to v_p$ and $v_p \to v_k$. Reversal of those two arcs in T yields an n-tournament with score sequence $\mathbf{s}$. On the other hand, if $\mathbf{s}$ is the score sequence of some n-tournament W in which vertex v_i has score s_i, $1 \leq i \leq n$, then we may suppose that $v_k \to v_{k+m-1}$ in W, for otherwise, interchanging the labels on v_k and v_{k+m-1} does not change $\mathbf{s}$. Reversal of the arc $v_k \to v_{k+m-1}$ in W yields an n-tournament with score sequence $\mathbf{s}'$. That is, $\mathbf{s}'$ is a score sequence if and only if $\mathbf{s}$ is a score sequence.

Next, we show that $\sum_{i=1}^{j} s_i > \binom{j}{2}$, $k \leq j \leq k+m-2$. Suppose, on the contrary, that for some j, $k \leq j < k+m-2$, $\sum_{i=1}^{j} s_i \leq \binom{j}{2}$. Conditions (1) imply that $\sum_{i=1}^{j} s_i \geq \binom{j}{2}$, so equality holds. Then, again by (1), $s_{j+1} + \binom{j}{2} = s_{j+1} + \sum_{i=1}^{j} s_i = \sum_{i=1}^{j+1} s_i \geq \binom{j+1}{2} = \binom{j}{2} + j$. So, $s_{j+1} \geq j$. As $s_j = s_{j+1}$, $s_j \geq j$. Thus, $\sum_{i=1}^{j} s_i = \sum_{i=1}^{j-1} s_i + s_j \geq \binom{j-1}{2} + s_j \geq \binom{j-1}{2} + j = \binom{j}{2} + 1 > \binom{j}{2}$, a contradiction to our supposition. So, $\sum_{i=1}^{j} s_i > \binom{j}{2}$, $k \leq j \leq k+m-2$.

Now we can show that **s** satisfies (1) if and only if **s′** satisfies (1). If **s** satisfies (1), then

$$\sum_{i=1}^{j} s_i' = \begin{cases} \sum_{i=1}^{j} s_i, & \text{if } j \le k-1; \\ \sum_{i=1}^{k-1} s_i + (s_k - 1) + \sum_{i=k+1}^{j} s_i, & \text{if } k \le j \le k+m-2; \\ \sum_{i=1}^{k-1} s_i + (s_k - 1) + \sum_{i=k+1}^{k+m-2} s_i + (s_{k+m-1} + 1) + \sum_{i=k+m}^{j} s_i, & \text{if } j \ge k+m-1. \end{cases}$$

In cases $j \le k-1$ and $j \ge k+m-1$, we see that $\sum_{i=1}^{j} s_i' = \sum_{i=1}^{j} s_i \ge \binom{j}{2}$. In cases $k \le j \le k+m-2$, the strict inequality established above implies that $\sum_{i=1}^{j} s_i' = (\sum_{i=1}^{j} s_i) - 1 > \binom{j}{2} - 1$. So, **s′** satisfies (1). On the other hand, if **s′** satisfies (1), then it is clear that **s** satisfies (1).

Let us define a total order on integer sequences that satisfy (1) as follows: $\mathbf{a} = (a_1, a_2, \ldots, a_n) \le \mathbf{b} = (b_1, b_2, \ldots, b_n)$ if either $\mathbf{a} = \mathbf{b}$, or $a_n < b_n$, or for some i, $1 \le i < n$, $a_n = b_n$, $a_{n-1} = b_{n-1}, \ldots, a_{i+1} = b_{i+1}$, $a_i < b_i$. Clearly, $\le$ is reflexive, antisymmetric, transitive, and satisfies comparability. Write $\mathbf{a} < \mathbf{b}$ if $\mathbf{a} \le \mathbf{b}$, but $\mathbf{a} \ne \mathbf{b}$. Note that, for any sequence $\mathbf{s} \ne \mathbf{t}_n$ satisfying (1), $\mathbf{s} < \mathbf{t}_n$, where $\mathbf{t}_n$ is the fixed sequence (0, 1, 2, . . . ,n-1), the score sequence for the transitive n-tournament. We have shown above that for every sequence $\mathbf{s} \ne \mathbf{t}_n$ satisfying (1) we can produce another sequence **s′** satisfying (1) such that $\mathbf{s} < \mathbf{s}'$. Moreover, **s** is a score sequence if and only if **s′** is a score sequence. So, by repeated application of this transformation starting from the original sequence satisfying (1) we must eventually reach $\mathbf{t}_n$. Thus, **s** is a score sequence, as required. ■

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