The Steiner Distance Dimension of Graphs

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Abstract

For a nonempty set S of vertices of a connected graph G, the Steiner distance $d(S)$ of S is the minimum size among all connected subgraphs whose vertex set contains S. For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the Steiner representation $s(v|W)$ of v with respect to W is the $(2^k - 1)$-vector

$$s(v|W) = (d_1(v), d_2(v), \cdots, d_k(v), d_{1,2}(v), d_{1,3}(v), \cdots, d_{1,2,\cdots,k}(v))$$

where $d_{i_1,i_2,\cdots,i_j}(v)$ is the Steiner distance $d(\{v, w_{i_1}, w_{i_2}, \cdots, w_{i_j}\})$. The set W is a Steiner resolving set for G if, for every pair u, v of distinct vertices of G, u and v have distinct representations. A Steiner resolving set containing a minimum number of vertices is called a Steiner basis for G. The cardinality of a Steiner basis is the Steiner (distance) dimension $\dim_S(G)$. In this paper, we study the Steiner dimension of graphs and determine the Steiner dimensions of several classes of graphs.

1 Introduction

A fundamental problem in chemistry is to represent a set of chemical compounds in such a way that distinct compounds have distinct representations. A graph-theoretic interpretation of this problem is to provide representations for the vertices of a graph in such a way that distinct vertices have distinct representations. For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices in a connected graph G and a vertex v of G, the k-vector (ordered k-tuple)

$$r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

is referred to as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if, for every pair u, v of distinct vertices of G, u and v have distinct representations. A resolving set containing a minimum number of vertices is called a *minimum resolving set* or a *basis* for G. The number of vertices in a basis

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for G is its *(metric) dimension* $\dim(G)$. This is the subject of the papers [1], [2], [3], and [4].

In this paper, we approach this problem from another point of view, namely, we use Steiner distance as a means of providing a refinement to representing the vertices of a graph. For a nonempty set S of vertices of a connected graph G, the *Steiner distance* $d(S)$ of S (or simply the distance of S) is the minimum size among all connected subgraphs whose vertex set contains S. If F is a connected subgraph of G such that $S \subseteq V(F)$ and $|E(F)| = d(S)$, then necessarily F is a tree, called a *Steiner tree* of S in G. If $S = \{u, v\}$, then $d(S) = d(u, v)$ and a Steiner tree of S is a $u - v$ path (indeed, a $u - v$ geodesic). If G has order n and $|S| = n$ (so $S = V(G)$), then $d(S) = n - 1$ and every spanning tree of G is a Steiner tree for S. For example, let $S = \{u, v, x\}$ in the graph G of Figure 1. Here $d(S) = 4$. There are several trees of size 4 containing S, one of which is the tree T of Figure 1.

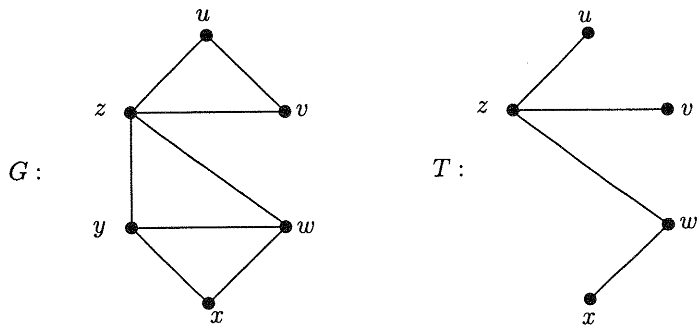

Figure 1: A graph G and a Steiner tree T

For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices in a connected graph G, and for $v \in V(G)$, the *Steiner representation* $s(v|W)$ of v with respect to W is the $(2^k - 1)$-vector

$$s(v|W) = (d_1(v), d_2(v), \cdots, d_k(v), d_{1,2}(v), d_{1,3}(v), \cdots, d_{1,2,\cdots,k}(v))$$

where $d_{i_1,i_2,\cdots,i_j}(v)$ is the Steiner distance $d(\{v, w_{i_1}, w_{i_2}, \cdots, w_{i_j}\})$. If, for every pair u, v of distinct vertices, u and v have distinct Steiner representations with respect to W, then W is a *Steiner resolving set* for G. A Steiner resolving set of minimum cardinality is called a *minimum Steiner resolving set* or a *Steiner basis* for G. The number of vertices in a Steiner basis is the *Steiner (distance) dimension* $\dim_S(G)$.

For each $v \in V(G)$, the first k coordinates in the Steiner representation $s(v|W)$ of v is the ordinary representation $r(v|W)$ of v with respect to W. Thus every resolving set for G is a Steiner resolving set for G, and so

$$\dim_S(G) \leq \dim(G) \tag{1}$$

To see that inequality (1) can be strict, we consider the graph G of Figure 2. We first show that $\dim_S(G) = 2$. Let $W = \{v_1, v_3\}$. The Steiner representations of the vertices of G with respect to W are

$$\begin{array}{lll} s(u_1 \mid W) = (1,3,4) & s(u_2 \mid W) = (2,2,4) & s(u_3 \mid W) = (3,1,4) \\ s(u_4 \mid W) = (4,2,5) & s(u_5 \mid W) = (3,3,6) & s(u_6 \mid W) = (2,4,5) \\ s(v_1 \mid W) = (0,4,4) & s(v_2 \mid W) = (3,3,5) & s(v_3 \mid W) = (4,0,4) \\ s(v_4 \mid W) = (5,3,6) & s(v_5 \mid W) = (4,4,7) & s(v_6 \mid W) = (3,5,6) \end{array}$$

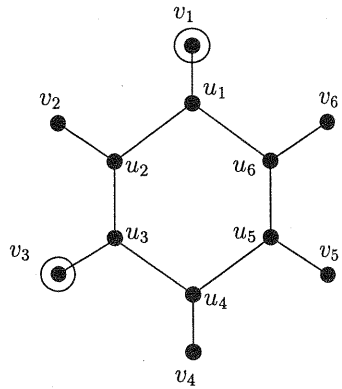

Figure 2: A graph G for which $\dim_S(G) < \dim(G)$

Since the representations are distinct, W is a Steiner resolving set for G. Certainly, no single vertex of G is a Steiner resolving set for G, and so $\dim_S(G) = 2$. It is straightforward to show that no 2-element set of vertices is a resolving set for G. Since the set $\{u_3, u_6, v_4\}$ is a resolving set, $\dim(G) = 3$.

It was shown in [1] that the dimension of a graph of order n and diameter d is at most $n - d$. So we have the following result.

Theorem 1.1 *If G is a connected graph of order $n \geq 2$ and diameter d, then*

$$\dim_S(G) \leq n - d$$

The upper bound in Theorem 1.1 is sharp. For example, the graph G of Figure 3 has order $n = 8$ and diameter $d = 4$, while $S = \{v_1, v_5, v_6, v_7\}$ is a Steiner basis for G and so $\dim_S(G) = 4$.

2 The Steiner Dimension of Certain Graphs

If G is a nontrivial connected graph, then certainly $1 \leq \dim_S(G) \leq n - 1$. For each $n \geq 2$, there is only one graph of order n having Steiner dimension 1.

Theorem 2.1 *A connected graph of order n has Steiner dimension 1 if and only if $G = P_n$.*

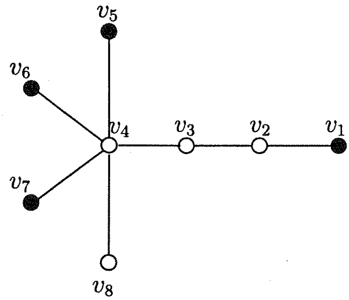

Figure 3: The graph G

Proof. We have already noted that if $G = P_n$, then $\dim_S(G) = 1$, as either end-vertex of G forms a Steiner resolving set for G. For the converse, assume that G is a connected graph of order n with $\dim_S(G) = 1$ and basis $W = \{w\}$. For each vertex v of G, $s(v|W) = d(v, w)$ is a nonnegative integer less then n. Since the representations of the vertices of G with respect to W are distinct, there exists a vertex u of G such that $d(u, w) = n - 1$. Consequently, the diameter of G is $n - 1$, which implies that $G = P_n$. ■

Theorem 2.2 *A connected graph G of order n has Steiner dimension $n-1$ if and only if $G = K_n$.*

Proof. First assume that G is a connected graph of order n such that $\dim_S(G) = n - 1$. Then $\dim(G) = n - 1$, which implies that $G = K_n$ [1]. Now we verify the converse. Assume, to the contrary, that there exists a Steiner resolving set W for $G = K_n$ which contains less than $n - 1$ vertices. Let x and y be two vertices in $V(G) - W$. Now for every k-subset of vertices from W, the Steiner distance from x to W is the same as the Steiner distance from y to W, for this distance is k, the smallest sized tree which can possibly contain x (respectively y) and all other vertices in the k-subset. We know that it is possible to obtain this tree of size k, since G is a complete graph. Therefore, the Steiner representation of x with respect to W is the same as the Steiner representation of y with respect to W. Therefore, $\dim_S(K_n) \geq n - 1$, so $\dim_S(K_n) = n - 1$. ■

In [1], it was shown that if G is a connected graph of order $n \geq 4$, then $\dim(G) = n - 2$ if and only if $G = K_{r,s}$, $G = K_s + \overline{K}_t$ where $t \geq 2$, or $G = K_s + (K_t \cup K_1)$, where $t \geq 2$. The next theorem states that it is precisely these graphs of order n for which the Steiner dimension equals $n - 2$.

Theorem 2.3 *Let G be a connected graph of order $n \geq 4$. Then $\dim_S(G) = n-2$ if and only if $G = K_{r,s}$ $(r, s \geq 1)$, or $K_s + \overline{K}_t$, $(s \geq 1, t \geq 2)$, or $K_s + (K_t \cup K_1)$, $(s, t \geq 1)$.*

Proof. Since $\dim_S(G) \leq \dim(G)$ for every connected graph G of order n, it follows that $\dim_S(G) \leq n - 2$ if $G = K_{r,s}$ $(r, s \geq 1)$, or $K_s + \overline{K}_t$, $(s \geq 1, t \geq 2)$, or

$K_s + (K_t \cup K_1)$, $(s, t \geq 1)$. Now we show that the Steiner dimension of each of these graphs is at least $n-2$.

We first consider $K_{r,s}$ $(r, s \geq 1)$, with bipartition $(X = \{x_1, x_2, \ldots, x_r\}, Y = \{y_1, y_2, \ldots, y_s\})$. Suppose, to the contrary, that there exists a Steiner resolving set W which contains at most $n-3$ vertices. Then there exists a pair of vertices $u, v \notin W$ which are both contained either in X or in Y. Suppose, without loss of generality, that u and v are both contained in X. Let $W_k = \{w_1, w_2, \ldots, w_k\}$ be a k-subset of vertices in W. Also suppose, without loss of generality, that $w_1, w_2, \ldots, w_i$ are contained in Y and that $w_{i+1}, w_{i+2}, \ldots, w_k$ are contained in X. Since G is a complete bipartite graph, we form two trees, one of which contains the edges $uw_1, uw_2, \ldots, uw_i, w_1w_{i+1}, w_1w_{i+2}, \ldots, w_1w_k$ and the second of which contains the edges $vw_1, vw_2, \ldots, vw_i, w_1w_{i+1}, w_1w_{i+2}, \ldots, w_1w_k$. Observe that each of these trees has size k. Furthermore, observe that there exists no other tree of smaller size which contains every vertex in $W_k \cup \{u\}$. Similarly, there is no other tree of smaller size which contains every vertex in $W_k \cup \{v\}$. This implies that the representation of u with respect to W is the same as the representation of v with respect to W, so W is not a Steiner resolving set for $K_{r,s}$. Therefore, $\dim_S(K_{r,s}) \geq n-2$.

Next we consider $G = K_s + \overline{K}_t$, $(s \geq 1, t \geq 2)$. Let $X = V(K_s) = \{x_1, x_2, \ldots, x_s\}$ and let $Y = V(\overline{K}_t) = \{y_1, y_2, \ldots, y_t\}$. Suppose, to the contrary, that there is a Steiner resolving set W which contains at most $n-3$ vertices. Then there exists two vertices u and v not in W which are either both contained in X or both contained in Y. Again, let $W_k = \{w_1, w_2, \ldots, w_k\}$ be a k-subset of vertices in W. Suppose, without loss of generality, that $w_1, w_2, \ldots, w_i$ are contained in Y and that $w_{i+1}, w_{i+2}, \ldots, w_k$ are contained in X.

Case 1.1: u and v are both contained in X. We form two trees, one of which contains the edges $uw_1, uw_2, \ldots, uw_k$, and the second of which contains the edges $vw_1, vw_2, \ldots, vw_k$. There exists no other tree of smaller size which contains every vertex in $W_k \cup \{u\}$, and there exists no other tree of smaller size which contains every vertex in $W_k \cup \{v\}$. Therefore, the representation of u with respect to W is the same as the representation of v with respect to W. So W is not a Steiner resolving set for G.

Case 1.2: u and v are both contained in Y. Again we form two trees, one of which contains the edges $uw_{i+1}, uw_{i+2}, \ldots, uw_k, w_1w_{i+1}, w_2w_{i+1}, \ldots, w_iw_{i+1}$ and the second of which contains the edges $vw_{i+1}, vw_{i+2}, \ldots, vw_k, w_1w_{i+1}, w_2w_{i+1}, \ldots, w_iw_{i+1}$. Once again there exists no other tree of smaller size which contains every vertex in $W_k \cup \{u\}$, and there exists no other tree of smaller size which contains every vertex in $W_k \cup \{v\}$. So the representation of u with respect to W is the same as the representation of v with respect to W. Therefore, W is not a Steiner resolving set for G.

Finally, we consider $G = K_s + (K_t \cup K_1)$, $(s, t \geq 1)$. Let $X = V(K_s) = \{x_1, x_2, \ldots, x_s\}$, let $Y = V(K_t) = \{y_1, y_2, \ldots, y_t\}$, and let $V(K_1) = \{z\}$. Suppose, to the contrary, that there is a Steiner resolving set W which contains at most $n-3$ vertices. We consider two cases.

Case 2.1: $z \in W$. If $z \in W$, then there exist two vertices u and v not in W which are either both contained in X or both contained in Y. Again, let $W_k = \{w_1, w_2, \ldots, w_k\}$ be a k-subset of vertices in W. Suppose, without loss of generality, that $w_1, w_2, \ldots, w_i$ are contained in Y, that $w_{i+1}, w_{i+2}, \ldots, w_{k-1}$ are contained in X, and that $z = w_k$. First suppose that u and v are both contained in X. Then we form two trees, one of which contains the edges $uw_1, uw_2, \ldots, uw_k$, and the second of which contains the edges $vw_1, vw_2, \ldots, vw_k$. So the representation of u with respect to W is the same as the representation of v with respect to W. Next suppose that u and v are both contained in Y. We again form two trees, one of which contains the edges $uw_1, uw_2, \ldots, uw_{k-1}, w_{i+1}w_k$, and the second of which contains the edges $vw_1, vw_2, \ldots, vw_{k-1}, w_{i+1}w_k$. So the representation of u with respect to W is the same as the representation of v with respect to W.

Case 2.2: $z \notin W$. Let $u, v \notin W$, where $u, v \neq z$. We consider two subcases.

Subcase 2.2.1. $u, v \in X$ or $u, v \in Y$. Then a similar argument to the one in Case 1 will show that $s(u|W) = s(v|W)$.

Subcase 2.2.2. One of u, v is in X, and one is in Y, say $u \in X$ and $v \in Y$. We show that $s(u|W) = s(v|W)$. We let $W_k = \{w_1, w_2, \ldots, w_k\}$ be a k-subset of vertices in W. We assume, without loss of generality that $w_1, w_2, \ldots, w_i$ are contained in Y and that $w_{i+1}, w_{i+2}, \ldots, w_k$ are contained in X. Then we form two trees, one of which contains the edges $uw_1, uw_2, \ldots, uw_k$, and the second of which contains the edges $vw_1, vw_2, \ldots, vw_k$. Certainly, the representation of u with respect to W is the same as the representation of v with respect to W. ■

Theorem 2.4 *The cycle of order $n \geq 3$ has Steiner dimension 2.*

Proof. Let $C_n : v_1, v_2, \cdots, v_n, v_1$. By Theorem 2.1, $\dim_S(C_n) \geq 2$. Since $W = \{v_1, v_2\}$ is a Steiner resolving set of C_n, it follows that $\dim_S(C_n) = 2$. ■

3 The Steiner Dimension of $L(K_n)$

We begin by presenting some preliminary concepts which will enable us to determine the Steiner dimension of the line graph of the complete graph of order n, $L(K_n)$.

The *distance* between an edge e_1 and an edge e_2, denoted $d_\epsilon(e_1, e_2)$, is the number of internal vertices on a shortest path which contains both e_1 and e_2.

Let G be a graph of order n and let $E(G)$ denote the set of edges of G. Let $X = \{e_1, e_2, \ldots, e_k\}$ be a set of edges in G. For each edge $e \in E(G)$, the distance representation of e with respect to X is the ordered k-tuple $r_\epsilon(e \mid X) = (d_\epsilon(e, e_1), d_\epsilon(e, e_2), \ldots, d_\epsilon(e, e_k))$. If, for every pair of edges f and g in G, $r_\epsilon(f \mid X) \neq r_\epsilon(g \mid X)$, then X is said to be an *edge resolving set* for G. An edge resolving set of minimum cardinality is called an *edge basis* for G. The cardinality of an edge basis for G is called the *edge dimension* of G and is denoted $\dim_\epsilon(G)$.

A vertex u in the line graph of K_n is distance 1 (respectively distance 2) from a vertex v if and only if the edge in K_n corresponding to u is distance 1 (respectively distance 2) from the edge in K_n corresponding to v. Therefore, if a set of edges in K_n forms an edge resolving set for K_n, then the set of vertices in $L(K_n)$ corresponding to these edges will form a resolving set for $L(K_n)$. Hence, it follows that the edge dimension of K_n is precisely the distance dimension of $L(K_n)$. Furthermore, since the Steiner dimension of any graph G, $\dim_S(G)$, is at most the dimension of G, it follows that $\dim_S(L(K_n)) \leq \dim_\epsilon(K_n)$, for all positive integers $n \geq 2$. In fact, we will show that if $n \geq 2$, $\dim_S(L(K_n)) = \dim_\epsilon(K_n)$.

We begin by determining the edge dimension of K_n.

Theorem 3.1 *For every integer $n \geq 3$,*

$$\dim_\epsilon(K_n) = \begin{cases} 2n/3 & \text{if } n \equiv 0 \pmod 3, \\ (2n+1)/3 & \text{if } n \equiv 1 \pmod 3 \\ (2n+2)/3 & \text{if } n \equiv 2 \pmod 3, n \neq 5 \\ 3 & \text{if } n = 5 \end{cases}$$

Proof. Let $V(K_n) = \{v_0, v_1, \ldots, v_{n-1}\}$. For $n \geq 3$, we begin by constructing an edge resolving set for K_n which contains $(2n+i)/3$ edges if $n \equiv i \pmod 3$ and $n \neq 5$, and for K_5 we construct an edge resolving set containing 3 edges. If $n \equiv 0 \pmod 3$, then for $0 \leq i \leq n/3 - 1$, let the edge resolving set X consist of the 2-paths $v_{3i}, v_{3i+1}, v_{3i+2}$; if $n \equiv 1 \pmod 3$, let X consist of the 2-paths $v_{3i}, v_{3i+1}, v_{3i+2}$ for $0 \leq i \leq (n-4)/3$ and the edge $v_{n-3}v_{n-1}$; if $n \equiv 2 \pmod 3$ and $n \neq 5$, then for $0 \leq i \leq (n-5)/3$ let X consist of the 2-paths $v_{3i}, v_{3i+1}, v_{3i+2}$ and the 2-path $v_{n-3}, v_{n-2}, v_{n-1}$; and if $n = 5$, let X consist of the 3-star containing the edges v_0v_1, v_1v_2, and v_1v_3. So X can be described as follows:

- a collection of 2-paths if $n \equiv 0 \pmod 3$;
- a collection of 2-paths and a 3-star if $n \equiv 1 \pmod 3$;
- a collection of 2-paths and a 4-path if $n \equiv 2 \pmod 3$ and $n \neq 5$; or
- a 3-star if $n = 5$.

We now show that X is an edge resolving set for K_n. It is easily verified that X is an edge resolving set for K_n, for $3 \leq n \leq 8$, and that a 4-path forms an edge resolving set for K_5. We now consider the case where $n \geq 9$. Let $G[X]$ denote the subgraph of K_n induced by the edges of X. Consider the edges $e, f \notin G[X]$. Suppose that all edges in $G[X]$ that are distance 1 from e lie in the component C_1 (or in the components C_1 and C_2) of $G[X]$. Then e and f could have the same representations with respect to X only if all edges in $G[X]$ that are distance 1 from f lie in the component C_1 (or in the components C_1 and C_2) of $G[X]$. But the subgraph of K_n induced by the vertices in C_1 is K_3, K_4, or K_5. Furthermore, the edges in C_1 which belong to X form an edge resolving set for C_1, so e and f have distinct representations. Similarly, the subgraph of K_n induced by the vertices in $C_1 \cup C_2$ is K_6, K_7, or K_8. The edges in $C_1 \cup C_2$ which belong to X form an edge resolving set for $C_1 \cup C_2$, so e and f have distinct representations with respect to X.

Suppose Y is an edge resolving set for K_n, where $|Y| \leq (2n+i)/3$ if $n \equiv i \pmod 3$, and let $G[Y]$ denote the subgraph of K_n induced by the edges in Y. First, $G[Y]$ cannot contain a component having only one edge y_1y_2. To show this we first consider $z \notin \{y_1, y_2\}$ which is a vertex of K_n. If $G[Y]$ contains a component having only the edge y_1y_2, then $r_\epsilon(zy_1 \mid Y) = r_\epsilon(zy_2 \mid Y)$. Furthermore, at least $n-2$

vertices of K_n must be incident with at least one edge in Y, for if two vertices, say z_1 and z_2, are not incident with an edge in Y, then the edges z_1y and z_2y will have the same representations with respect to Y, for each $y \in V(K_n) - \{z_1, z_2\}$. In addition, if there exists some vertex in K_n which is not incident with at least one edge of Y, then no component of $G[Y]$ can contain less than 3 edges. It has already been established that $G[Y]$ cannot contain any component of size one, so suppose that $G[Y]$ contains some component C having exactly two edges, say wx and xy, and that there exists some vertex z which is not incident with any edge of Y. Then zx and wy have the same edge representations with respect to Y. So if exactly $n-1$ vertices are incident with at least one edge of Y, then each of these vertices must be contained in a component which has at least 3 edges.

Having established some characteristics of edge resolving sets for K_n, we now show that the edge resolving set X constructed previously is indeed an edge basis for K_n.

First suppose that $n = 3x$ (so X contains $2x$ edges), and let Y be an edge resolving set containing $2x - j$ edges, for some positive integer $j < 2x$. Again, we denote by $G[Y]$ the subgraph of K_n induced by the edges of Y. We assume, without loss of generality, that each component of $G[Y]$ has minimum possible size with respect to the order (that is, each component is a tree). Suppose that $G[Y]$ has k nontrivial components of sizes $c_1, c_2, \ldots, c_k$. Then $c_1 + c_2 + \cdots + c_k = 2x - j$. Since each component of $G[Y]$ is a tree, it follows that the number of vertices in $G[Y]$ is $c_1 + c_2 + \ldots + c_k + k = 2x - j + k$. Therefore, if $G[Y]$ contains all vertices of K_n, then $2x - j + k = 3x$, so $k = x + j$. It was established earlier that each of these $x + j$ components must have size at least 2, so this means that $G[Y]$ contains at least $2x + 2j$ edges, which is a contradiction. If $G[Y]$ contains $3x - 1$ vertices, then $2x - j + k = 3x - 1$, so $k = x + j - 1 \geq x$. It was established earlier that if $G[Y]$ contains an isolated vertex, then each nontrivial component must contain at least 3 edges, so it follows that $G[Y]$ contains at least $3x$ edges, which is a contradiction.

Next suppose that $n = 3x + 1$ (so X contains $2x + 1$ edges), and let Y be an edge resolving set containing $2x + 1 - j$ edges, for some positive integer $j < 2x + 1$. Assume once again that $G[Y]$ consists of k nontrivial components of sizes $c_1, c_2, \ldots, c_k$. Then $c_1 + c_2 + \cdots + c_k = 2x + 1 - j$. Now $G[Y]$ contains $c_1 + c_2 + \cdots + c_k + k = 2x + k - j + 1$ vertices. If $G[Y]$ contains every vertex of K_n, then $2x + k - j + 1 = 3x + 1$, so $k = x + j$. Since each component contains at least 2 edges, $G[Y]$ must contain at least $2x + 2j$ edges, which is a contradiction. Now we suppose that $G[Y]$ contains $3x$ vertices. Therefore, $2x + k - j + 1 = 3x$, so $k = x + j - 1$. Since $G[Y]$ does not contain all $3x + 1$ vertices, each component of $G[Y]$ must contain at least 3 edges, so it follows that $G[Y]$ contains at least $3(x + j - 1)$ edges, which is a contradiction.

Finally suppose that $n = 3x + 2$ (so X contains $2x + 2$ edges), and let Y be an edge resolving set containing $2x + 2 - j$ edges, for some positive integer $j < 2x + 2$. We assume that $G[Y]$ consists of k nontrivial components of sizes $c_1, c_2, \ldots, c_k$. Then $c_1 + c_2 + \cdots + c_k = 2x + 2 - j$. Now $G[Y]$ contains $c_1 + c_2 + \cdots + c_k + k = 2x + k - j + 2$ vertices. If $G[Y]$ contains every vertex of K_n, then $2x + k - j + 2 = 3x + 2$, so $k = x + j$. Each component of $G[Y]$ has size at least 2, so the number of edges in $G[Y]$ is at least $2x + 2j$, which is a contradiction. Now suppose that $G[Y]$ contains $3x + 1$ vertices.

Then $2x + k - j + 2 = 3x + 1$, so $k = x + j - 1$. Therefore, if each component must contain at least 3 edges, it follows that $G[Y]$ contains at least $3x + 3j - 3$ edges, which is a contradiction as long as $x > 1$ or $j > 1$. However, if $x = 1$ and $j = 1$, it follows that $n = 5$, and there exists an edge basis containing $2x + 2 - j = 3$ edges.

Therefore, the edges of X form an edge basis for K_n. ■

Corollary 3.2 *Let $n \geq 3$ be a positive integer. Then*

$$\dim(L(K_n)) = \begin{cases} 2n/3 & \text{if } n \equiv 0 \pmod 3, \\ (2n+1)/3 & \text{if } n \equiv 1 \pmod 3 \\ (2n+2)/3 & \text{if } n \equiv 2 \pmod 3, n \neq 5 \\ 3 & \text{if } n = 5 \end{cases}$$

Certainly, for any connected graph G, $\dim_S(G) \leq \dim(G)$. In what follows, we show that we have equality if $G = L(K_n)$.

Theorem 3.3 *Let $n \geq 2$ be a positive integer. Then* $\dim_S(L(K_n)) = \dim(L(K_n))$.

Proof. Certainly, if $n = 2$, then $\dim_S(L(K_n)) = \dim(L(K_n))$, so we assume $n \geq 3$.

Let $G = L(K_n)$. We first assume that $\dim_S(G) < \dim(G)$ and work toward a contradiction. Let S be a Steiner basis for G. Since $\dim_S(G) < \dim(G)$, it follows that there exist two vertices $x, y \in V(G)$ such that $r(x \mid S) = r(y \mid S)$, but $s(x \mid S) \neq s(y \mid S)$. We define S_i, $i \in \{1, 2\}$, to be the set of vertices in S which are distance i from x and y. There are two cases to consider.

Case 1: x and y are nonadjacent.

If x and y are nonadjacent vertices, then we consider a partition of $V(G) - \{x, y\}$. Let X_i (respectively Y_i), $i \in \{1, 2\}$, denote the set of vertices in $V(G) - \{x, y\}$ which are distance i from vertex x (respectively, vertex y). Consider the edges $x' = ab$ and $y' = cd$ in K_n which correspond to vertices x and y, respectively, in G. The only edges in K_n which are distance 1 from both x' and y' are ac, ad, bc, and bd, so it follows that $|X_1 \cap Y_1| = 4$. Furthermore, the edge x' is incident with $2(n-4)$ edges besides ac, ad, bc, and bd, and the edge y' is also incident with $2(n-4)$ edges besides these four. So $|X_1 - (X_1 \cap Y_1)| = |Y_1 - (X_1 \cap Y_1)| = 2(n-4)$ (and the sets $X_1 - (X_1 \cap Y_1)$ and $Y_1 - (X_1 \cap Y_1)$ have empty intersection). Also, the only edges in K_n which are distance 2 from both x' and y' are those in the clique induced by $V(K_n) - \{a, b, c, d\}$, so $|X_2 \cap Y_2| = (n-4)(n-5)/2$. We observe that $X_2 - (X_2 \cap Y_2) = Y_1 - (X_1 \cap Y_1)$ and $Y_2 - (X_2 \cap Y_2) = X_1 - (X_1 \cap Y_1)$ (see Figure 4).

Since $r(x \mid S) = r(y \mid S)$, it follows that $S \subseteq (X_1 \cap Y_1) \cup (X_2 \cap Y_2)$. Let T be a Steiner tree which contains $\{x\} \cup S$. Then T must contain some minimum nonempty subset $\Gamma = \{v_1, v_2, \ldots, v_k\}$ of vertices in $X_1 - (X_1 \cap Y_1)$. Otherwise, if $V(T)$ is entirely contained in $(X_1 \cap Y_1) \cup (X_2 \cap Y_2)$, then certainly T can be modified to produce a tree T^* with size $|E(T)|$ which contains $\{y\} \cup S$; T^* can be formed from T by replacing all edges of the form xv_i, $v_1 \in X_1 \cap Y_1$, with yv_i, where $1 \leq i \leq k$.

Each vertex in Γ is adjacent to some set of vertices in $X_2 \cap Y_2$ which are also in the Steiner basis S; in particular, for each $v_i \in \Gamma$, there is a subset $A =$

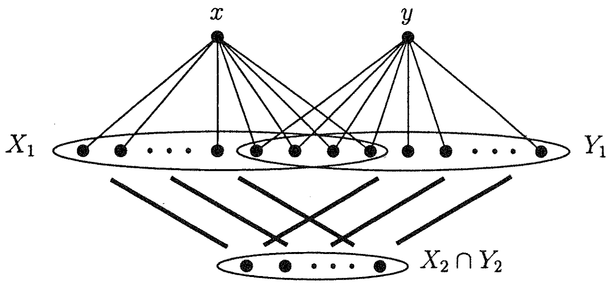

Figure 4: A partition of $V(L(K_n))$ when x and y are not adjacent.

$\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}\} \subseteq S$, each of whose vertices is adjacent to v_i in T. However, for each $v_i \in \Gamma$, there exists some vertex $w_i \in Y_1 - (X_1 \cap Y_1)$ which is adjacent to each vertex in A. This can be seen more clearly by considering the edges in K_n which correspond to vertices in A and in Γ. The vertex $v_i \in \Gamma$ corresponds to some edge, say $v_i' = ae$, in K_n (recall that $x' = ab$, so we assume without loss of generality that vertex a is incident with both x' and v_i'). Now each edge in K_n which corresponds to a vertex in A must be incident with vertex $e \in V(K_n)$, for if any such edge is incident with vertex a, then this would imply that some vertex in A is contained in X_1. Furthermore, there exists some edge of the form $w_i' = ce$ which is distance 1 from y' and which is distance 1 from the edges corresponding to the vertices in A. Since such an edge exists, there exists a corresponding vertex $w_i \in Y_1 - (X_1 \cap Y_1)$ which is distance 1 from the vertices in A. So for each $v_i \in \Gamma$ there is a vertex w_i which is adjacent to the same vertices in $(X_2 \cap Y_2)$ to which v_i is adjacent. This implies that we can build a tree T^* which has size $|E(T)|$ and which contains $\{y\} \cup S$. Therefore, $s(x \mid S) = s(y \mid S)$, which is a contradiction.

Case 2: x and y are adjacent.

The proof is similar to the proof of Case 1. The differences lie in the fact that $|X_1 \cap Y_1| = n - 2$, $|X_1 - (X_1 \cap Y_1)| = |Y_1 - (X_1 \cap Y_1)| = n - 3$, and $|X_2 \cap Y_2| = (n-3)(n-4)/2$. Otherwise, the proof is essentially identical.

Therefore, $\dim_S(G) = \dim(G)$. ∎

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