ON THE NORDHAUS-GADDUM CLASS PROBLEMS

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Abstract: Let $\mathcal{G}(n)$ denote the class of simple graphs of order n. For $G \varepsilon \mathcal{G}(n)$, $\bar{G}$ denotes the complement of G. Given a graph theoretic parameter f, the **Nordhaus-Gaddum Problem** is to find lower and upper bounds for:

$$f(G) + f(\bar{G}) ,$$

and

$$f(G) . f(\bar{G}) ,$$

over the class $\mathcal{G}(n)$.

In this paper we consider a variation of this problem by restricting our attention to the subclass of $\mathcal{G}(n)$ consisting of graphs having exactly m edges. We consider the parameters edge connectivity, diameter and chromatic number. We also consider the problem of characterizing the extremal graphs and the realizability problem.

1. INTRODUCTION

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part, our notation and terminology follow that of Bondy and Murty [6]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges, edge connectivity $\kappa'(G)$, chromatic number $\chi(G)$, maximum degree $\Delta(G)$ and minimum degree $\delta(G)$. However, we denote the complement of the graph G by $\bar{G}$ and the diameter of G by d(G).

Let $\mathcal{G}(n)$ denote the class of graphs of order n and $\mathcal{G}(n,m)$ the subclass having m edges. Given a graph theoretic parameter f(G) and a positive integer n, the **Nordhaus-Gaddum (N-G) Problem** is to determine sharp bounds for:

(i) $f(G) + f(\overline{G})$,

and

(ii) $f(G) \,.\, f(\overline{G})$,

as G ranges over the class $\mathcal{G}(n)$, and characterize the extremal graphs. A further problem is to determine the set of all integer pairs (x,y) such that $f(G) = x$ and $f(\overline{G}) = y$ for some $G \in \mathcal{G}(n)$. We refer to this latter problem as the **realizability problem**.

In their paper, Nordhaus-Gaddum [12] determined bounds for $\chi(G) + \chi(\overline{G})$ and $\chi(G).\chi(\overline{G})$. The characterization of the corresponding extremal graphs and the realizability problem were resolved by Finck [10]. Since this early work a number of authors have considered other graph theoretic parameters such as: edge chromatic number (Alavi and Behzad [2], Vizing [15]); total chromatic number (Achuthan [1]); clique number (Chartrand and Schuster [8]); edge connectivity (Alavi and Mitchem [3]); and diameter (Bondy [5]).

A number of variations to the N-G problem have been considered - Dirac [9], Plesník [14]. In this paper, we consider a further variation. Our variation considers the above mentioned problems when G is restricted to the subclass $\mathcal{G}(n,m)$. We are motivated to consider this variation as many of the sharp bounds in the classical N-G problem are attained by one of the graphs G and $\overline{G}$ being very dense. We present results for the parameters: diameter, edge connectivity and chromatic number.

2. DIAMETER

Since the diameter of a disconnected graph is ∞, we shall in this section restrict ourselves to the case when G and $\overline{G}$ are connected. Hence $d(G) \geq 2$ and $d(\overline{G}) \geq 2$. The Nordhaus-Gaddum bounds for this class of graphs are summarized in the following theorem.

Theorem 2.1 Let G be a connected graph on $n \geq 6$ vertices with a connected complement $\overline{G}$. Then

$$4 \le d(G) + d(\overline{G}) \le n + 1 , \qquad (2.1)$$

and

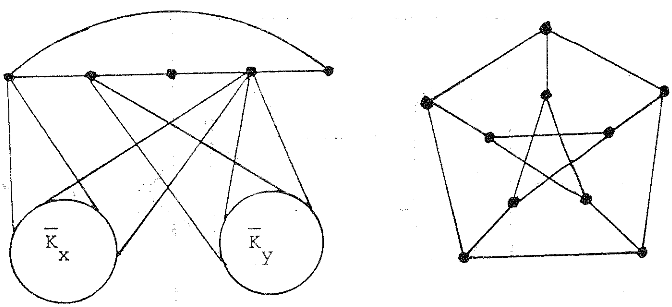
$$4 \le d(G).d(\overline{G}) \le 2(n - 1) . \qquad (2.2)$$

Moreover these bounds are sharp. □

The upper bound in (2.1) was established by Bosák et.al. [7] and also by Bondy [5]. The bounds in (2.2) are immediate consequences of the fact that if $d(G) \ge 4$ then $d(\overline{G}) \le 2$.

Our objective in this section is to consider the functions $d(G) + d(\overline{G})$ and $d(G).d(\overline{G})$ when $G \in \mathcal{G}(n,m)$. We can assume that our graph G has diameter 2 or 3. With each of these cases we shall consider, separately, the cases when $\overline{G}$ has diameter 2,3 and ≥ 4. We find it convenient to specify the values of $d(G)$ and $d(\overline{G})$ and then determine the possible values of m. This, of course, solves the above mentioned problem. In our discussion we make use of the following results from the literature which we state as lemmas.

Lemma 2.1 (Murty [11]).
Let $G \in \mathcal{G}(n,m)$ be a 2-connected graph of diameter 2. Then $m \ge 2n - 5$ with equality possible only if G is one of the two graphs drawn in Figure 2.1. □



$x + y = n - 5,\ x \ge 0,\ y \ge 0$

Figure 2.1

A tree of diameter 3 whose centre is a pair of adjacent vertices is called a **double star** (see Figure 2.2). Observe that the complement of a double star has diameter 3.

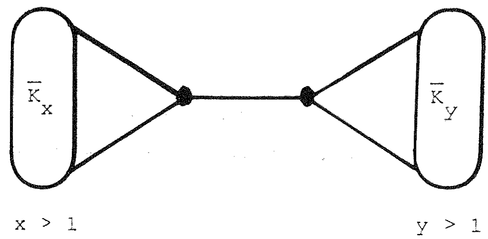


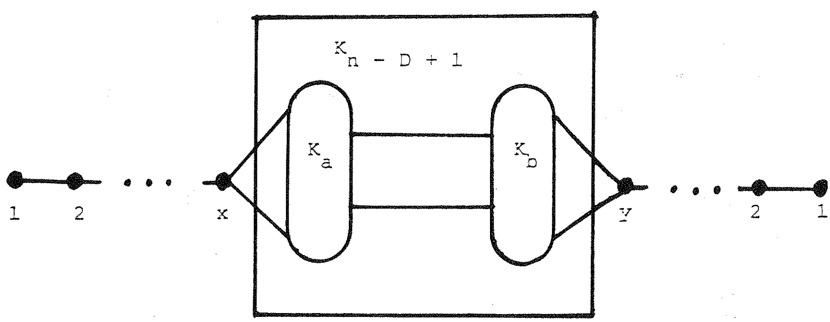
Figure 2.2 Double star

Lemma 2.2 (Bloom et.al. [4])
A graph G has diameter 2 if and only if $\overline{G}$ is non-empty and $\overline{G}$ is not spanned by a double star. □

Lemma 2.3 (Ore [13])
Let $G \in \mathcal{G}(n,m)$ be a graph of diameter $D \geq 4$. Then

$$m \leq 2n - D - 2 + \frac{1}{2}(n - D)(n - D - 1)$$

with equality possible only if G is the graph in Figure 2.3. □



$x + y = D - 1,\ x \geq 1,\ y \geq 1,\ a \geq 1,\ b \geq 1.$

Figure 2.3

Theorem 2.2 Let $G \in \mathcal{G}(n,m)$, $n \geq 5$, be a graph of diameter 2 whose complement $\overline{G}$ has diameter 2. Then

$$2n - 5 \leq m \leq \frac{1}{2}n(n - 1) - (2n - 5) . \qquad (2.3)$$

Furthermore, every integer in the above range is realizable.

Proof: If G has a cut vertex, say x, then $d_G(x) = n - 1$ and hence $\overline{G}$ cannot be connected. Therefore G is 2-connected. Similarly $\overline{G}$ is 2-connected. Hence, by Lemma 2.1, G and $\overline{G}$ each have at least $2n - 5$ edges with equality possible. This proves (2.3).

Consider the graph G drawn below in which H is a graph on $n - 4$ vertices each vertex of which is adjacent to u and v. Clearly G and $\overline{G}$ have diameter 2 and $\varepsilon(G) = 2n - 5 + \varepsilon(H)$.

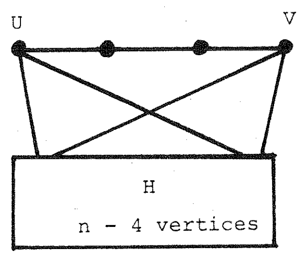


Figure 2.4

with $0 \leq \varepsilon(H) \leq \frac{1}{2}(n - 4)(n - 5)$. Hence every integer m satisfying (2.3) is realizable. □

Consider the graphs drawn in Figure 2.5 over.

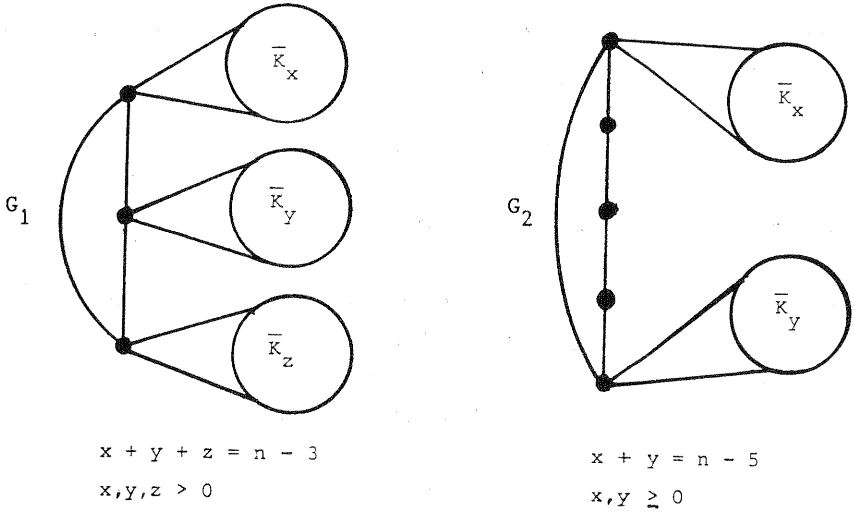


Figure 2.5

Observe that $d(G_i) = 3$ and $d(\bar{G}_i) = 2$, $i = 1,2$. Also $\varepsilon(G_i) = n$. We now prove that these graphs are edge-minimal among the class $G \in \mathcal{G}(n,m)$ for which $d(G) = 3$ and $d(\bar{G}) = 2$.

Lemma 2.4 Let $G \in \mathcal{G}(n,m)$, $n \geq 6$, be a graph of diameter 3 whose complement has diameter 2. Then $m \geq n$ and this bound is achievable for every n. Further, apart from two exceptions the extremal graphs are those of Figure 2.5. The two exceptional graphs are the cycles C_n for n = 6 and 7.

Proof: Suppose that $m < n$. Then G must be a tree of diameter 3. That is, G is a double star. But then, by Lemma 2.2 $d(\bar{G}) \neq 2$. Hence $m \geq n$. The graphs in Figure 2.5 show that this bound is sharp for $n \geq 6$. To establish the structure of the extremal graphs we prove that for $n \geq 6$ an edge-minimal G contains a cycle of length 3 or 5.

Let G be an edge-minimal graph. Clearly G is unicyclic. Let c be the length of the cycle in G. Then $c \leq 5$ (except when $G \cong C_6$ or C_7) as otherwise G has diameter > 3. So suppose c = 4. Let C = u, v, w, x, u denote the cycle of length 4. Now the (n - 4)-vertices of

G − C must each be joined to exactly one vertex of C, as otherwise $\varepsilon(G) > n$. In fact, since $d(G) = 3$, no non-adjacent pairs of vertices of C can have neighbours in $V(G - C)$. But then G has a spanning double star and hence $d(\bar{G}) > 2$. So $c \neq 4$; it must therefore be 3 or 5. Now it is only a simple exercise to verify that $G = G_1$ when $c = 3$ and $G = G_2$ when $c = 5$. For $n \leq 7$ the G can also consist of the cycle C_n. This completes the proof of the lemma. □

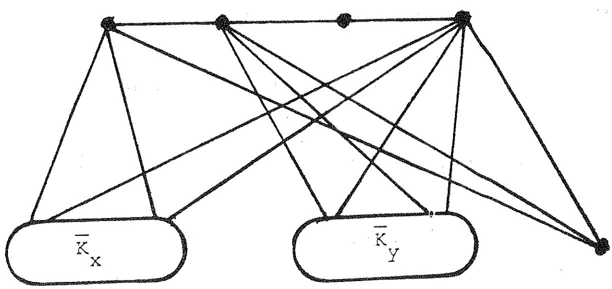
Theorem 2.3 Let $G \in \mathcal{G}(n,m)$ be a graph of diameter 3 whose complement $\bar{G}$ has diameter 2. Then for $n \geq 6$

$$n \leq m \leq \frac{1}{2}n(n-1) - 2(n-2) . \qquad (2.4)$$

Furthermore, every integer in the above range is realizable.

Proof: The lower bound in (2.4) was established in Lemma 2.4. Consider $\bar{G}$. If $\bar{G}$ has a cut vertex then G is disconnected. Hence $\bar{G}$ is 2-connected and so, by Lemma 2.1, $\varepsilon(\bar{G}) \geq 2n - 5$. Now suppose that $\varepsilon(\bar{G}) = 2n - 5$. Then by Lemma 2.1, $\bar{G}$ is one of the graphs in Figure 2.1. Examination of the graphs in Figure 2.1 reveals that their complements have diameter 2. This contradicts the fact that the diameter of G is 3. Hence $\varepsilon(\bar{G}) \geq 2n - 4$. This establishes the upper bound in (2.4).

In the following we explain the realizability of integers in the range given in (2.4).



$x + y = n - 5,\quad x > 0,\ y \geq 0$

Figure 2.6

For an integer m between n and $\frac{1}{2}(n - 4)(n - 5) + 5$, we start with the graph G_1 in Figure 2.5 with $x = y = 1$ and add $(m - n)$ edges suitably.

For an integer m between $\frac{1}{2}n(n - 1) - \frac{1}{2}(n - 5)(n - 6) - (2n - 4)$ and $\frac{1}{2}n(n - 1) - (2n - 4)$, we start with the complement of the graph in Figure 2.6 with $y = 0$ and delete $\frac{1}{2}n(n - 1) - (2n - 4) - m$ edges suitably.

This procedure results in a graph $G \in \mathcal{G}(n,m)$ with $d(G) = 3$ and $d(\bar{G}) = 2$, for all possible integers m except when (i) $n = 7$, $m = 9$ and (ii) $n = 8$, $m = 12$. In these exceptional cases one can easily check the realizability. This completes the proof of the theorem. □

Theorem 2.4 Let $G \in \mathcal{G}(n,m)$ be a graph of diameter 3 whose complement has diameter 3. Then

$$n - 1 \le m \le \frac{1}{2}n(n - 1) - (n - 1)$$

and every integer in the above range is realizable.

Proof: The bounds follow from the fact that both G and $\bar{G}$ are connected. To establish the sharpness let G be the graph displayed in Figure 2.2. The realizability can be easily established by starting with the same graph and adding edges suitably.

This completes the proof of Theorem 2.4. □

Theorem 2.5 Let $G \in \mathcal{G}(n,m)$ be a graph of diameter $D \ge 4$. Then

$$n - 1 \le m \le 2n - D - 2 + \frac{1}{2}(n - D)(n - D - 1), \qquad (2.5)$$

and every integer in this range is realizable.

Proof: The lower bound in (2.5) is obvious, the upper bound follows from Lemma 2.3. The realizability can be established by starting with a suitable tree with diameter D and adding m - (n - 1) edges in such a way that the diameter remains D.

This completes the proof of the theorem. □

3. EDGE CONNECTIVITY

The edge connectivity $\kappa'(G)$ of a graph G is the minimum number of edges whose removal results in a disconnected graph. The N-G bounds are given in the following result.

Theorem 3.1 (Alavi and Mitchem [3])

Let $G \in \mathcal{G}(n)$. Then

$$1 \le \kappa'(G) + \kappa'(\bar{G}) \le n - 1,$$

and

$$0 \le \kappa'(G).\kappa'(\bar{G}) \le M(n),$$

$$\text{where} \quad M(n) = \begin{cases} \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil, & \text{if } n = 0,1,2 \pmod 4 \\ \left(\frac{n-3}{2}\right)\left(\frac{n+1}{2}\right), & \text{otherwise.} \end{cases}$$

□

In our discussion we make use of the following basic facts:

(i) For any graph G

$$\kappa'(G) \le \delta(G), \tag{3.1}$$

and

$$\kappa'(G) \le \frac{2\varepsilon(G)}{\nu(G)} . \tag{3.2}$$

(ii) If $\delta(G) \ge \frac{1}{2}(\nu(G) - 1)$, then $\kappa'(G) = \delta(G)$.

(iii) If $G \in \mathcal{G}(n)$ is a graph such that $\kappa'(G) + \kappa'(\bar{G}) = n - 1$, then G is regular of degree $\kappa'(G)$.

(iv) For every n and d such that $2 \le d \le n-1$ there exists a $G \in \mathcal{G}(n, \lceil \frac{1}{2}nd \rceil)$ such that $\kappa'(G) = d$.

The extremal graphs for which (iv) holds are due to Harary and are described in standard graph theory texts. In our constructions to be described we often modify these graphs. Hence it is convenient to describe the Harary graphs, denoted by $H(n,d)$.

We shall give the construction separately for even and odd values of d.

First let d be even and $d = 2r$. The vertex set of $H(n,d)$ is $\{0,1,2,\ldots,n-1\}$. The vertex i is joined to $i \pm j$, $1 \le j \le r$ where addition is modulo n.

Next let d be odd and $d = 2r + 1$. When n is even, say $n = 2n_1$, we construct $H(n,2r)$ as explained in the case when d is even and further join i and $i + n_1$ for $0 \le i \le n_1 - 1$. When n is odd, say $n = 2n_1 + 1$ we start with $H(n, d-1)$ and further join 0 to vertices n_1 and $n_1 + 1$ and vertex i to vertex $i + n_1 + 1$ for $1 \le i < n_1$.

For the rest of this section, without any loss of generality we assume that $m \le \frac{1}{2}\binom{n}{2}$.

Lemma 3.1 Let $G \in \mathcal{G}(n,m)$. Then

$$\kappa'(G) + \kappa'(\bar{G}) \ge \max\{1,\ n-1-m\} \tag{3.3}$$

and

$$\kappa'(G).\kappa'(\bar{G}) \ge 0. \tag{3.4}$$

These bounds are sharp for all n and m.

Proof: The inequality (3.4) follows from Theorem 3.1. Since at least one of G and $\bar{G}$ must be connected, we have,

$$\kappa'(G) + \kappa'(\bar{G}) \ge 1 \tag{3.5}$$

The inequality (3.3) reduces to (3.5) when $n - 1 \le m$. So let $n - 1 > m$. We shall show that

$$\kappa'(G) + \kappa'(\bar{G}) \ge n - 1 - m . \qquad (3.6)$$

Since $n - 1 > m$, G is disconnected and so $\kappa'(G) = 0$.

Now (3.6) reduces to $\kappa'(\bar{G}) \ge n - 1 - m$.

If possible let $\kappa'(\bar{G}) < n - 1 - m$ and E a cutset of edges of $\bar{G}$ with $|E| = \kappa'(\bar{G})$. Let $H_1, H_2, \ldots, H_k$ $(k \ge 2)$ be the connected components of $\bar{G} - E$ and $\nu(H_i) = n_i$ for $1 \le i \le k$. Then

$$\varepsilon(\bar{G} - E) \le \sum_{i=1}^{k} \binom{n_i}{2} \le \binom{n-1}{2} .$$

Therefore,

$$\varepsilon(\bar{G}) \le \kappa'(\bar{G}) + \binom{n-1}{2} < n - 1 - m + \binom{n-1}{2} = \binom{n}{2} - m ,$$

which contradicts the fact that $G \in \mathcal{G}(n,m)$. Hence (3.6) is established and this proves (3.3). To establish the sharpness of the bounds, we construct a graph $G^* \in \mathcal{G}(n,m)$ as follows:

Case 1: $m < n - 1$. Define $V(G^*) = \{u_1, u_2, \ldots, u_n\}$ and $E(G^*) = \{(u_i, u_n),\ 1 \le i \le m\}$. It is easy to see that $\kappa'(G^*) = 0$ and $\kappa'(\bar{G}^*) = n - 1 - m$.

Case 2: $m \ge n - 1$. Let $V(G^*) = \{u_1, u_2, \ldots, u_n\}$. We define $E(G^*)$ as follows:

(a) u_1 is adjacent to u_i, for $2 \le i \le n$.

(b) the remaining $m - (n - 1)$ edges are introduced on the set $\{u_2, u_3, \ldots u_{n-1}\}$.

It is easy to see that $\kappa'(G^*) = 1$ and $\kappa'(\bar{G}^*) = 0$. This establishes the sharpness of the bounds. □

Lemma 3.2 Let $G \in \mathcal{G}(n,m)$. Then

$$\kappa'(G) + \kappa'(\bar{G}) \le n - 1, \tag{3.7}$$

and

$$\kappa'(G).\kappa'(\bar{G}) \le M(n,m), \tag{3.8}$$

where

$$M(n,m) = \begin{cases} 0\,, & \text{if } m \le n-2 \\ \frac{2m}{n}\left(n - 1 - \frac{2m}{n}\right), & \text{if } 2m \equiv 0 \pmod n \text{ and } m \ge n \\ \left\lfloor \frac{2m}{n} \right\rfloor \left(n - 2 - \left\lfloor \frac{2m}{n} \right\rfloor\right), & \text{otherwise.} \end{cases}$$

These bounds are sharp whenever $2m \equiv 0 \pmod n$ with the exception that when $m = \frac{n}{2}$ we have

$$\kappa'(G) + \kappa'(\bar{G}) \le n - 2$$

and this bound is sharp.

Proof: The inequality (3.7) follows from Theorem 3.1. To establish (3.8) first let $m \le n - 2$. For $G \in \mathcal{G}(n,m)$, clearly

$$\kappa'(G) = 0. \tag{3.9}$$

Next let $2m \equiv 0 \pmod n$, say $2m = nd$. By (3.2) we have

$$\kappa'(\bar{G}) \le \frac{2\varepsilon(\bar{G})}{n} = n - 1 - d \tag{3.10}$$

Finally let $2m = nd + \ell$, where $1 \le \ell \le n - 1$. Now, $\Delta(G) \ge d + 1$ and so

$$\delta(\bar{G}) = n - 1 - \Delta(G) \le n - 2 - d.$$

By (3.1) we have

$$\kappa'(\overline{G}) \le n - 2 - d. \tag{3.11}$$

Combining (3.2), (3.9), (3.10) and (3.11) we have the inequality (3.8). To prove the sharpness of the bounds once again let $2m = nd$.

Case d = 0. We have $m = 0$ and thus $G = \overline{K}_n$ and $\overline{G} = K_n$. Clearly in this case the bounds in (3.7) and (3.8) are sharp.

Case d = 1. Now $m = \frac{n}{2}$ and in this exceptional case, clearly $\kappa'(G) = 0$ and $\kappa'(\overline{G}) \le n - 2$. Thus

$$\kappa'(G).\kappa'(\overline{G}) = 0 \quad \text{and} \quad \kappa'(G) + \kappa'(\overline{G}) \le n - 2.$$

To establish the sharpness of this bound let G^* be a set of m independent edges.

Case d ≥ 2. Let $G^* = H(n,d)$.

By Fact (iv)

$$\kappa'(G^*) = d \tag{3.12}$$

We will now show that $\kappa'(\overline{G}^*) = n - 1 - d$. Since $m \le \frac{1}{2}\binom{n}{2}$, we have $d \le \frac{n-1}{2}$. Therefore $\delta(\overline{G}^*) = n - 1 - d \ge \frac{n-1}{2}$ and now using Fact (ii) we have

$$\kappa'(\overline{G}^*) = \delta(\overline{G}^*) = n - 1 - d \tag{3.13}$$

Now combining (3.12) and (3.13) we establish the sharpness of the upper bounds in (3.7) and (3.8). This completes the proof of Lemma 3.2. □

Lemma 3.3 Let $G \in \mathcal{G}(n,m)$ and $2m \not\equiv 0 \pmod n$. Then

$$\kappa'(G) + \kappa'(\overline{G}) \le N(n,m), \tag{3.14}$$

and

$$\kappa'(G).\kappa'(\overline{G}) \le M(n,m), \tag{3.15}$$

where $$N(n,m) = \begin{cases} n - 3, & \text{if } n + 1 \le 2m \le 2n - 4 \\ n - 2, & \text{otherwise} \end{cases}$$

and $M(n,m)$ is as defined in Lemma 3.2. These bounds are sharp for all m and n.

Proof: Using the fact that $2m \not\equiv 0 \pmod{n}$ and Fact (iii) we notice that

$$\kappa'(G) + \kappa'(\bar{G}) \le n - 2 \tag{3.16}$$

The inequality (3.15) follows from Lemma 3.2.

The proof of the sharpness is divided into different cases.

Case A: m ≥ n. Let $2m = nd + \ell$ where $1 \le \ell \le n - 1$.

Subcase (i): d is even. Let G^* be the graph $H(n,d)$ along with the edges $(i, i + \lfloor \frac{n}{2} \rfloor)$ for $0 \le i \le \frac{\ell - 2}{2}$. Clearly $\delta(G^*) = d$ and $\Delta(G^*) = d + 1$. It is easy to show that

$$\kappa'(G^*) = d. \tag{3.17}$$

We shall prove that

$$\kappa'(\bar{G}^*) = n - 2 - d. \tag{3.18}$$

From the assumption that $m \le \frac{1}{2}\binom{n}{2}$ and the fact that $\ell \ge 1$ it follows that $d \le \frac{1}{2}(n - 2)$.

If $d < \frac{n-2}{2}$, then $\delta(\bar{G}^*) = n - 2 - d \ge \frac{n-1}{2}$. Now using Fact (ii), we have

$$\kappa'(\bar{G}^*) = \delta(\bar{G}^*) = n - 2 - d.$$

Next let $d = \frac{n-2}{2}$. Using the structure of G^* we shall show that,

$$\kappa'(\bar{G}^*) = n - 2 - d = \frac{n-2}{2} .$$

If possible let $\kappa'(\bar{G}^*) \le \frac{n-4}{2}$ and E a set of edges such that $|E| = \kappa'(\bar{G}^*)$ and whose removal disconnects $\bar{G}^*$. Then there exists a subset A $\subseteq V(\bar{G}^*)$ such that the set of edges between A and $V(\bar{G}^*) - A$ is precisely E.

Without loss of generality let $|A| \le |V(\bar{G}^*) - A|$ and $t = |A|$. Then it follows that $t \le \frac{n}{2}$. Now

$$\sum_{x \in A} d_{\bar{G}^*}(x) \le t(t-1) + \frac{n-4}{2} \tag{3.19}$$

and

$$\sum_{x \in A} d_{\bar{G}^*}(x) \ge \left[\frac{n-2}{2} \right] t . \tag{3.20}$$

Combining (3.19) and (3.20) and rearranging the terms we have

$$(t-1)(t - \frac{n-2}{2}) - 1 \ge 0 .$$

This is possible only if $t \ge \frac{n}{2}$.

This gives $t = \frac{n}{2}$ and it follows that there exist $x_1, x_2 \in A$ such that they are not incident with any of the edges in E. Thus x_1 and x_2 are adjacent to all the vertices of V - A in G^* and are not adjacent to any vertex in A. This is a contradiction since from the construction of G^*, we can see that there is no such pair x_1, x_2 with exactly the same neighbour sets. Thus $\kappa'(\bar{G}^*) = \frac{n-2}{2}$, and this establishes (3.18). Using (3.17) and (3.18), the sharpness of (3.15) and (3.16) is verified in this subcase.

Subcase (ii): d is odd; n is even. Note that ℓ is even in this case. Let G^* be the graph H(n,d) along with the edges $(i, \frac{n-2}{2} + i)$ for $0 \le i \le \frac{\ell - 2}{2}$. Now by arguing along the same lines as in subcase (i) we can show the sharpness of (3.15) and (3.16) in this subcase.

Subcase (iii): n and d are both odd. In this case ℓ is also odd. Let G^* be the graph $H(n,d)$ along with $\frac{\ell - 1}{2}$ new edges added in such a way that $\Delta(G^*) = d + 1$. Clearly $\kappa'(G^*) = d$. In this case also one can easily show that $\kappa'(\bar{G}^*) = n - 2 - d$, thus establishing the sharpness of (3.15) and (3.16).

Case B: $1 \le m \le n - 1$.
Subcase (i): $1 \le m < \frac{n}{2}$. Let $G^* \in \mathcal{G}(n,m)$ be such that $\Delta(G^*) \le 1$. Note that G^* is disconnected and so $\kappa'(G^*) = 0$. Now $\kappa'(\bar{G}^*) = n - 2$. This establishes the sharpness of (3.15) and (3.16) in this subcase.

Subcase (ii): $\frac{n}{2} < m \le n - 2$. In this case it is easy to see that

$$\kappa'(G) + \kappa'(\bar{G}) \le n - 3 \qquad (3.21)$$

To show that this bound is sharp, we let G^* be a path on $m + 1$ vertices. Clearly $\kappa'(G^*) = 0$ and $\kappa'(\bar{G}^*) = n - 3$. When $m \le n - 2$, obviously G is disconnected and hence $\kappa'(G) = 0$ giving $\kappa'(G).\kappa'(\bar{G}) = 0$.

Subcase (iii): $m = n - 1$. In this case take G^* to be a path on n vertices. Then $\kappa'(G^*) = 1$ and $\kappa'(\bar{G}^*) = n - 3$. This establishes the sharpness of the upper bounds in (3.15) and (3.16) in this subcase. This completes the proof of the lemma. □

Combining Lemmas 3.1, 3.2 and 3.3 we state the following theorem.

Theorem 3.2 Let $G \in \mathcal{G}(n,m)$. Then

$$\max\{1, n - 1 - m\} \le \kappa'(G) + \kappa'(\bar{G}) \le N(n,m), \qquad (3.22)$$

and

$$0 \le \kappa'(G).\kappa'(\bar{G}) \le M(n,m), \qquad (3.23)$$

where

$$N(n,m) = \begin{cases} n-3, & \text{if } n+1 \le 2m \le 2n-4 \\ n-2, & \text{if } (2 \le 2m \le n) \text{ or } (m = n-1) \\ & \text{or } (2m \not\equiv 0 \pmod n) \text{ and } m \ge n). \\ n-1, & \text{otherwise} \end{cases}$$

and

$$M(n,m) = \begin{cases} 0, & \text{if } m \le n-2 \\ (\frac{2m}{n})(n-1-\frac{2m}{n}), & \text{if } 2m \equiv 0 \pmod n \text{ and } m \ge n \\ \left\lfloor \frac{2m}{n} \right\rfloor (n-2-\left\lfloor \frac{2m}{n} \right\rfloor), & \text{otherwise} \end{cases}$$

The bounds in (3.22) and (3.23) are sharp for all n and m. □

4. CHROMATIC NUMBER

The chromatic number $\chi(G)$ of a graph G is the minimum number of colours needed to colour the vertices of G such that no two adjacent vertices receive the same colour. We state some known results that we need for our discussion.

Theorem 4.1 (Nordhaus and Gaddum [12])

Let $G \in \mathcal{G}(n)$. Then

$$2\sqrt{n} \le \chi(G) + \chi(\bar{G}) \le n + 1$$

$$n \le \chi(G).\chi(\bar{G}) \le \left[\frac{n+1}{2} \right]^2.$$

These bounds are sharp for every n. □

Finck [10] characterized the extremal graphs for Theorem 4.1. We now describe some of these graphs which we make use of in our discussion.

For $1 \le \alpha \le n$, let $G_1(\alpha) = (K_1 \vee K_{\alpha-1}) \cup \bar{K}_{n-\alpha}$. Let $\mathcal{G}_1(n,m,\alpha)$ denote the class of graphs obtained from $G_1(\alpha)$ by adding $m - \frac{1}{2}\alpha(\alpha-1)$ edges between the vertices of $K_{\alpha-1}$ and $\bar{K}_{n-\alpha}$. Figure 4.1 illustrates the construction. Observe that if $G \in \mathcal{G}_1(n,m,\alpha)$ then $\chi(G) = \alpha$ and $\chi(\bar{G}) = n + 1 - \alpha$.

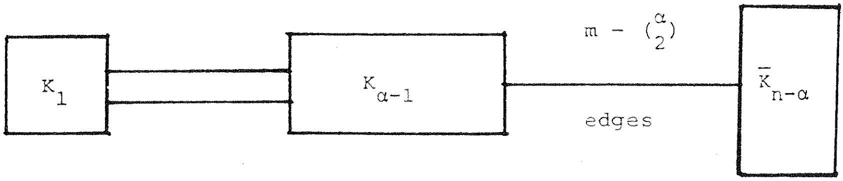


Figure 4.1 $\mathcal{G}_1(n, m, \alpha)$

For $1 \le \beta \le n - 5$, let $G_2(\beta) = (C_5 \vee K_\beta) \cup \overline{K}_{n-\beta-5}$. Let $\mathcal{G}_2(n, m, \beta)$ denote the class of graphs obtained from $G_2(\beta)$ by adding $m + 5 - \frac{1}{2}(\beta + 5)(\beta + 4)$ edges between the vertices of K_β and $\overline{K}_{n-\beta-5}$. Figure 4.2 illustrates the construction. Observe that if $G \in \mathcal{G}_2(n, m, \beta)$, then $\chi(G) = \beta + 3$ and $\chi(\overline{G}) = n - \beta - 2$.

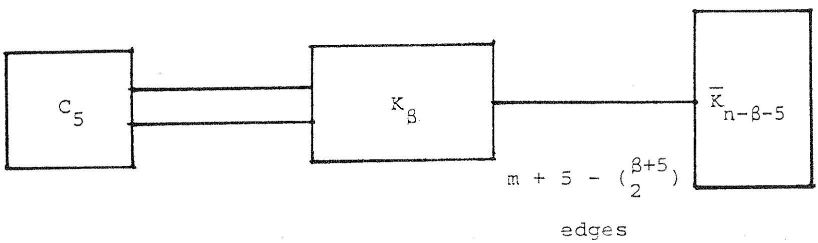


Figure 4.2 $\mathcal{G}_2(n, m, \alpha)$

Suppose d is a divisor of n. Let $G_3(d)$ denote the graph consisting of n/d disjoint copies of K_d. Label the vertices of the ith copy $v_1^{\,i}, v_2^{\,i}, \ldots, v_d^{\,i}$. Let $\mathcal{G}_3(n, m, d)$ denote the class of graphs obtained from $G_3(d)$ by adding $m - \frac{1}{2}n(d - 1)$ edges of the type $v_t^{\,i} v_{t'}^{\,j}$ such that $i \ne j$ and $t \ne t'$. Observe that for $G \in \mathcal{G}_3(n, m, d)$, the induced subgraph $G[v_t^{\,1}, v_t^{\,2}, \ldots, v_t^{\,n/d}]$ has no edges. Further, $\chi(G) = d$ and $\chi(\overline{G}) = n/d$.

We can now restate a result of Finck [10].

Lemma 4.1 Let $G \in \mathcal{G}(n)$. Then

(i) $\chi(G) + \chi(\bar{G}) = n + 1$ if and only if $G \in \mathcal{G}_1(n,m,\alpha) \cup \mathcal{G}_2(n,m,\beta)$ for some α, β and m;

(ii) $\chi(G).\chi(\bar{G}) = \left\lfloor \frac{1}{2}(n + 1)^2 \right\rfloor$ if and only if G or $\bar{G}$ belongs to $\mathcal{G}_1(n, m, \left\lceil \frac{n}{2} \right\rceil) \cup \mathcal{G}_2(n, m, \left\lceil \frac{1}{2}(n - 6) \right\rceil)$ for some m;

(iii) $\chi(G).\chi(\bar{G}) = n$ if and only if $G \in \mathcal{G}_3(n,m,d)$ for some divisor d of n and $m \geq \frac{1}{2}n(d - 1)$. □

Henceforth, we assume without any loss of generality that $m \leq \frac{1}{2}\binom{n}{2}$. Let ω be an integer such that $m = \binom{\omega}{2} + t$ where $0 \leq t \leq \omega - 1$.

Lemma 4.2 Let $G \in \mathcal{G}(n,m)$. Then

$$\chi(G) + \chi(\bar{G}) \leq n + 1 \ . \tag{4.1}$$

This bound is sharp for all n and m.

Proof: The inequality (4.1) follows from Theorem 4.1. Taking $G^* \in \mathcal{G}_1(n,m,\omega)$ the sharpness follows from Lemma 4.1 □

Lemma 4.3 Let $G \in \mathcal{G}(n,m)$, $n' = \left\lceil \frac{n}{2} \right\rceil$ and $m = \binom{\omega}{2} + t$, $0 \leq t \leq \omega - 1$. Then

$$\chi(G).\chi(\bar{G}) \leq A(n,m) \tag{4.2}$$

where

$$A(n,m) = \begin{cases} \left\lfloor \left(\frac{n + 1}{2}\right)^2 \right\rfloor , & \text{if } m \geq \binom{n'}{2} \\ \omega(n + 1 - \omega), & \text{otherwise} . \end{cases}$$

This bound is sharp for all n and m.

Proof: Let $m \geq \binom{n'}{2}$. Then the inequality (4.2) follows from Theorem 4.1 and the sharpness follows from Lemma 4.1. Now let $m < \binom{n'}{2}$. It is easy to see that $\chi(G) \leq \omega$ and consequently $\chi(G).\chi(\overline{G}) \leq \omega(n + 1 - \omega)$. To establish the sharpness in this case, define $G^* \in \mathcal{G}_1(n,m,\omega)$. This completes the proof of the lemma. □

Lemma 4.4 Let $G \in \mathcal{G}(n,m)$ and $n_1(\neq 1)$ be the smallest divisor of n. Then

$$\chi(G).\chi(\overline{G}) \geq B(n,m) \qquad (4.3)$$

where

$$B(n,m) = \begin{cases} n, & \text{if } m=0 \text{ or } n \text{ is not a prime and } m \geq \dfrac{(n_1-1)n}{2} \\ 2(n-m), & \text{if } 1 \leq m < \left\lceil \dfrac{n}{2} \right\rceil \\ n+1, & \text{otherwise.} \end{cases}$$

The inequality (4.3) is sharp for all n and m.

Proof: We shall divide the proof into three cases.

Case 1: $m = 0$ or n is not a prime and $m \geq \dfrac{(n_1 - 1)n}{2}$.

The inequality (4.3) follows from Theorem 4.1. To establish the sharpness, we define G^* as follows:

(i) $G^* \in \mathcal{G}_3(n,0,1)$, if $m = 0$;

(ii) $G^* \in \mathcal{G}_3(n,m,n_1)$, if $m \geq 1$. (This is always possible since $m \leq \frac{1}{2}\binom{n}{2}$).

Now invoking Lemma 4.1 the sharpness is established.

Case 2: n is odd, not a prime and $\left\lceil \frac{n}{2} \right\rceil \leq m < \dfrac{(n_1-1)n}{2}$ or n is a prime and $m \geq \left\lceil \frac{n}{2} \right\rceil$.

From Finck's characterization (Lemma 4.1), it follows that in this case

$$\chi(G).\chi(\overline{G}) \geq n + 1.$$

To establish the sharpness, let G^* be a subgraph of $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ such that G^* has a matching of size $\lfloor \frac{n}{2} \rfloor$ and $\varepsilon(G^*) = m$. This is possible since $m \geq \lceil \frac{n}{2} \rceil$. Clearly $\chi(G^*) = 2$ and $\chi(\overline{G^*}) = \frac{n+1}{2}$. Hence the sharpness follows.

Case 3: $1 \leq m < \lceil \frac{n}{2} \rceil$.

Since $m \geq 1$ we have $\chi(G) \geq 2$ and it is easy to see that $\chi(\overline{G}) \geq n - m$. Hence $\chi(G).\chi(\overline{G}) \geq 2(n - m)$. To establish the sharpness we take G^* to consist of m independent edges. Clearly $\chi(G^*) = 2$ and $\chi(\overline{G^*}) = n - m$. This establishes the sharpness and completes the proof of the lemma. □

Lemma 4.5 Let $G \in \mathcal{G}(n,m)$ and $n' = \lceil \frac{n}{2} \rceil$. Then

$$\chi(G) + \chi(\overline{G}) \geq C(n,m) \tag{4.4}$$

where

$$C(n,m) = \begin{cases} n + 1\ , & \text{if } m = 0 \\ n + 2 - m\ , & \text{if } 1 \leq m < n' \\ \lceil 2\sqrt{n} \rceil\ , & \text{otherwise}\ . \end{cases}$$

This inequality is sharp when $m < n'$.

Proof: If $m = 0$ then $G \cong \overline{K}_n$ and $\chi(G) = 0$ and $\chi(\overline{G}) = n$. If $1 \leq m < n'$, it is easy to see that $\chi(\overline{G}) \geq n - m$ and hence $\chi(G).\chi(\overline{G}) \geq n + 2 - m$ and the graph G^* consisting of m independent edges shows that the inequality is sharp. When $m \geq n'$ the inequality follows from Theorem 4.1. □

...ing Lemmas 4.1 through 4.3 we have the following theorem.

Theorem 4.2 Let $G \in \mathcal{G}(n,m)$, $n' = \left\lceil \frac{n}{2} \right\rceil$, $n_1 (\neq 1)$ be the smallest divisor of n and $m = \binom{\omega}{2} + t$, $0 \le t \le \omega - 1$.

Then

$$C(n,m) \le \chi(G) + \chi(\bar{G}) \le n + 1 \qquad (4.5)$$

and

$$B(n,m) \le \chi(G) . \chi(\bar{G}) \le A(n,m) \qquad (4.6)$$

where

$$A(n,m) = \begin{cases} \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor, & \text{if } m \ge \binom{n'}{2} \\ \omega(n + 1 - \omega), & \text{otherwise;} \end{cases}$$

$$B(n,m) = \begin{cases} n, & \text{if } m=0 \text{ or } n \text{ is not a prime and } m \ge \frac{(n_1-1)n}{2} \\ 2(n-m), & \text{if } 1 \le m < n' \\ n + 1, & \text{otherwise;} \end{cases}$$

and

$$C(n,m) = \begin{cases} n + 1, & \text{if } m = 0 \\ n + 2 - m, & \text{if } 1 \le m < n' \\ \left\lceil 2\sqrt{n} \right\rceil, & \text{otherwise.} \end{cases}$$

The upper bounds and the lower bound in (4.6) are sharp for all n and m. The lower bound in (4.5) is sharp for $0 \le m < n'$. □

The sharpness of the lower bound in (4.5) is not established for $m \ge n'$.

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