Blocking sets in balanced path designs*

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Abstract

Let $k \geq 3$. For each admissible v, we determine the set $\mathcal{BSH}(v,k,1)$ of integers x such that there exists a balanced path design $H(v,k,1)$ with a blocking set of cardinality x.

1 Introduction

Let G be a subgraph of K_v, the complete undirected graph on v vertices. A *G-design* of K_v is a pair $(V,\mathcal{B})$, where V is the vertex set of K_v and $\mathcal{B}$ is an edge-disjoint decomposition of K_v into copies of the graph G. Usually we say that b is a block of the G-design if $b \in \mathcal{B}$, and $\mathcal{B}$ is called the block-set. A G-design of K_v is also called a *G-design of order v*.

A *balanced G-design* [5, 4] is a G-design such that each vertex belongs to the same number of copies of G. Obviously not every G-design is balanced.

A *balanced path design* $H(v,k,1)$ [4] is a balanced P_k-design of K_v, where P_k is the simple path with $k-1$ edges (k vertices) $(a_1,a_2,\ldots,a_k) =$ $\{\{a_1,a_2\},\{a_2,a_3\},\ldots,\{a_{k-1},a_k\}\}$.

S. H. Y. Hung and N. S. Mendelsohn [5] proved that a $H(v,2h+1,1)$ $(h \geq 1)$ exists if and only if $v \equiv 1 \pmod{4h}$, and a $H(v,2h,1)$ $(h \geq 2)$ exists if and only if $v \equiv 1 \pmod{2h-1}$.

Given a $H(v,k,1)$ $(V,\mathcal{B})$, a subset X of V is called a *blocking set* of $\mathcal{B}$ if for each $b \in B$, $b \cap X \neq \emptyset$, and $b \cap (V-X) \neq \emptyset$. A $H(v,k,1)$ with blocking set is said to be *2-colorable*, and the partition $(X, V-X)$ is called a *2-coloring*.

Numerous articles have been written on the existence of blocking sets in projective spaces, in t-designs and in G-designs [1, 2, 7, 8, 9].

For each admissible v, let $\mathcal{BSH}(v,k,1)$ be the set of integers x such that there exists a $H(v,k,1)$ with a blocking set of cardinality x. S. Milici [8] determined $\mathcal{BSH}(v,k,1)$ for $k = 3,4$. The aim of this note is to determine $\mathcal{BSH}(v,k,1)$ for every $k \geq 3$.

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Theorem 1 (Necessary condition). *Let $x \in \mathcal{BSH}(v,k,1)$, then*

$$\frac{v-1}{k-1} \le x \le \frac{(k-2)v+1}{k-1}.$$

Proof. Let X be a blocking set in a $H(v,k,1)$ $(V,\mathcal{B})$, $|X|=x$. Since each $b \in \mathcal{B}$ meets X, we have $x\frac{k(v-1)}{2(k-1)} - \frac{x(x-1)}{2} \ge \frac{v(v-1)}{2(k-1)}$. This inequality and the fact that $V-X$ is a blocking set imply the proof. □

2 $\mathcal{BSH}(v,k,1)$ for even $k \ge 4$.

In this section we determine the set $\mathcal{BSH}(v,k,1)$ for each even $k \ge 4$. To prove this result we will use the $v \to v+k-1$ construction for $H(v,k,1)$ [5].

Lemma 1 *Let $k \ge 4$ be an even integer and let $x \in \mathcal{BSH}(v,k)$. Then $x+t \in \mathcal{BSH}(v+k-1,k,1)$ for each $t = 1,2,\dots,k-2$.*

Proof. Let $W = \{a_\mu \mid \mu = 0,1,\dots,k-1\}$. For each $i = 0,1,\dots,\frac{k}{2}-1$ and $\rho = 0,1,\dots,\frac{k}{2}-1$ let $y^i_{2\rho} = a_{\rho+i}$ and $y^i_{2\rho+1} = a_{k-1-\rho+i}$, where the indices $\rho+i$ and $k-1-\rho+i$ are reduced $\pmod k$ to the range $\{0,\dots,k-1\}$. Define the $H(k,k,1)$ $(W,\mathcal{D})$ by putting in D the blocks $(y^i_0, y^i_1, \dots, y^i_{k-1})$.

Put $v = 1+(k-1)\alpha$, $\alpha \ge 1$. Let $(V,\mathcal{B})$, $V \cap W = \emptyset$, be a $H(v,k,1)$ with a blocking set X. Let $V = (\bigcup_{t=1}^{k-1} X^t) \cup \{a_0\}$, $X^t = \{x^t_j \mid j = 1,2,\dots,\alpha\}$. Suppose $X^1 \subseteq X$ and $X^{k-1} \subseteq V - X$.

For each $j = 1,2,\dots,\alpha$, $i = 0,1,\dots,k-2$ and $\rho = 0,1,\dots,\frac{k}{2}-1$ put $z^{j,i}_{2\rho} = x^{k-\rho+i}_j$ and $z^{j,i}_{2\rho+1} = a_{1+\rho+i}$ where the indices $k-\rho+i$ and $1+\rho+i$ are reduced $\pmod{k-1}$ to the range $\{1,2,\dots,k-1\}$. Let $\mathcal{C}$ contain the blocks $(z^{j,i}_0, z^{j,i}_1, \dots, z^{j,i}_{k-1})$.

Put $\mathcal{E} = \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$. It is easy to verify that (see [5]) $(V \cup W, \mathcal{E})$ is a $H(v+k-1,k,1)$.

Now we prove that $\overline{X} = X \cup \{a_i \mid i = 1,2,\dots,t\}$, $t = 1,2,\dots,k-2$, is a blocking set of $\mathcal{E}$. Let $b \in \mathcal{E}$. If $b \in \mathcal{D}$ then $b \cap \overline{X} = \{a_i \mid i = 1,2,\dots,t\}$ and $b \cap (V - \overline{X}) = \{a_0\} \cup \{a_i \mid i = t+1, t+2, \dots, k-1\}$. If $b \in \mathcal{B}$ then $b \cap X \ne \emptyset$ and $b \cap (V-X) \ne \emptyset$. If $b \in \mathcal{C}$ then $b \cap (X^1 \cup \{a_1\}) \ne \emptyset$ and $b \cap (X^{k-1} \cup \{a_{k-1}\}) \ne \emptyset$. □

Theorem 2 *For each even $k \ge 4$ and for each $v \equiv 1 \pmod{k-1}$, we have* $\mathcal{BSH}(v,k,1) = \left\{ x \;\middle|\; \frac{v-1}{k-1} \le x \le \frac{(k-2)v+1}{k-1} \right\}.$

Proof. For $v = k$ the proof follows from the fact that each block has cardinality k. Theorem 1 and Lemma 1 complete the proof. □

3 $\mathcal{BSH}(v,k,1)$ for odd $k \geq 3$.

In this section we determine the set $\mathcal{BSH}(v,k,1)$ for each odd $k \geq 3$. We will use the difference method to construct $H(v,k,1)$ [3, 6].

Lemma 2 *If* $2 \in \mathcal{BSH}(4h+1, 2h+1, 1)$, $h \geq 1$, *then* $\mathcal{BSH}(4h+1,2h+1,1) = \{2,3,\ldots,4h-1\}$.

Proof. For $h = 1$ the proof is straight forward. Suppose $h \geq 2$. By Theorem 1 it is sufficient to prove that $\{2,3,\ldots,2h\} \subseteq \mathcal{BSH}(4h+1,2h+1,1)$.

Let X be a blocking set in a $\mathcal{BSH}(4h+1,2h+1,1)$ $(V,\mathcal{B})$, $|X| = 2$. For each x with $3 \leq x \leq 2h$, let Y be a subset of V such that $|Y| = x-2$ and $|Y \cap X| = 0$. Then $X \cup Y$ is a blocking set of $\mathcal{B}$. □

Theorem 3 *For each odd* $k \geq 3$ *and for each* $v \equiv 1 \pmod{k-1}$, *we have* $\mathcal{BSH}(v,k,1) = \left\{ x \;\middle|\; \frac{v-1}{k-1} \leq x \leq \frac{(k-2)v+1}{k-1} \right\}$.

Proof. Put $k = 2h+1$ and $v = 1+4hm$, $h \geq 1$ and $m \geq 1$. For each $j = 0,1,\ldots,m-1$ and $t = 0,1,\ldots,h-1$ define $a_0 = 0$, $a_{2t+1} = t+1+2hj$ and $a_{2t+2} = 4hm - t$. Using the difference method construct a $H(1+4hm, 1+2h, 1)$ $(V,\mathcal{B})$ having the following base blocks [6]:

$$b_j = (a_0, a_1, \ldots, a_{2h-1}, a_{2h}).$$

The *difference* of the pair $\{a_{i_1}, a_{i_2}\}$, named so that $a_{i_1} < a_{i_2}$, is defined to be $D(a_{i_1}, a_{i_2}) = \min\{a_{i_2} - a_{i_1}, v - (a_{i_2} - a_{i_1})\}$.

Let S be the set of the differences of the pairs $\{a_{i_1}, a_{i_2}\}$ where a_{i_1} and a_{i_2} are vertices of b_j, $j = 0,1,\ldots,m-1$, such that $\{a_{i_1}, a_{i_2}\}$ is not an edge in b_j. The elements of S are the following:
$D(a_{2t+2}, a_{2\rho}) = t - \rho + 1$ for each $\rho = 0,1,\ldots,h-1$ and $t = \rho+1, \rho+2, \ldots, h-1$;
and, if $h \geq 2$,
$D(a_{2\rho+1}, a_{2t+1}) = t - \rho$,
$D(a_{2\rho+1}, a_{2t+2}) = t + 2 + \rho + 2hj$,
$D(a_{2t+1}, a_{2\rho}) = t + 1 + \rho + 2hj$,
for each $\rho = 0,1,\ldots,h-2$ and $t = \rho+1, \rho+2, \ldots, h-1$.

It is easy to see that $S \cap \{2h\sigma \mid \sigma = 1,2,\ldots,2m-1\} = \emptyset$. So there is exactly one $b \in \mathcal{B}$ meeting both the elements $2h\sigma_1$ and $2h\sigma_2$, for every $\sigma_1, \sigma_2 \in \{0,1,\ldots,2m-1\}$, $\sigma_1 \neq \sigma_2$. Therefore the pair $\{2h\sigma_1, 2h\sigma_2\}$ is an edge in b.

Since every point of V meets $m(2h+1)$ paths of $\mathcal{B}$, the following inequalities hold:

$$1 \leq |b \cap \{2h\sigma \mid \sigma = 0,1,\ldots,2m-1\}| \leq 2 \quad \forall b \in \mathcal{B}. \tag{1}$$

From (1) it follows that $\{2h\sigma \mid \sigma = 0,1,\ldots,2m-1\}$ is a blocking set of minimum cardinality.

Put $X_i = \{i + 2h\sigma \mid \sigma = 0,1,\ldots,2m-1\}$, $i = 0,1,\ldots,h-1$. It is easy to see that $X_i \cap X_j = \emptyset$ for each $i,j \in \{0,1,\ldots,h-1\}$, $i \neq j$. Then from (1) and $|b| = 2h+1$

it follows that $\bigcup_{i=0}^{\mu} X_i$, $\mu = 1, 2, \ldots, h-1$, is a blocking set of $\mathcal{B}$ having cardinality $2m(\mu+1)$.

To complete the proof it is sufficient to prove that if x is an integer such that $2m < x < 2hm$ and $x \neq 2\mu m$ for each $\mu = 1, 2, \ldots, h-1$, then $x \in \mathcal{BSH}(1+4hm, 1+2h)$. For $m = 1$ this result follows from Lemma 2. For $m \geq 2$, let $x = 2\mu m + \sigma$, $\sigma = 1, 2, \ldots, 2m-1$. Then $(\bigcup_{i=0}^{\mu-1} X_i) \cup \{\mu + 2hj \mid j = 0, 1, \ldots, \sigma - 1\}$ is a blocking set having cardinality $2\mu m + \sigma$. □

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