Bicyclic Antiautomorphisms of Directed Triple Systems with 0 or 1 Fixed Points

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Abstract

A transitive triple, (a, b, c), is defined to be the set $\{(a, b), (b, c), (a, c)\}$ of ordered pairs. A directed triple system of order v, DTS(v), is a pair (D, β), where D is a set of v points and β is a collection of transitive triples of pairwise distinct points of D such that any ordered pair of distinct points of D is contained in precisely one transitive triple of β. An antiautomorphism of a directed triple system, (D, β), is a permutation of D which maps β to β^{-1}, where $\beta^{-1} = \{(c, b, a)|(a, b, c) \in \beta\}$. In this paper we give necessary and sufficient conditions for the existence of a directed triple system of order v admitting an antiautomorphism consisting of two cycles of equal length and having 0 or 1 fixed points.

1 PRELIMINARIES

A *Steiner triple system of order* v, STS(v), is a pair (S, β), where S is a set of v points and β is a collection of 3-element subsets of S, called *blocks*, such that any pair of distinct points of S is contained in precisely one block of β. Kirkman [4] showed that there is an STS(v) if and only if $v \equiv 1$ or 3 (mod 6) or $v = 0$.

A *transitive triple*, (a, b, c), is defined to be the set $\{(a, b), (b, c), (a, c)\}$ of ordered pairs. A *directed triple system of order* v, DTS(v), is a pair (D, β), where D is a set of v points and β is a collection of transitive triples of pairwise distinct points of D, called *triples*, such that any ordered pair of distinct points of D is contained in precisely one element of β. Hung and Mendelsohn [2] have shown that necessary and sufficient conditions for the existence of a DTS(v) are that $v \equiv 0$ or 1 (mod 3).

For a DTS(v), (D, β), we define β^{-1} by $\beta^{-1} = \{(c, b, a)|(a, b, c) \in \beta\}$. Then (D, β^{-1}) is a DTS(v) and is called the *converse* of (D, β). A DTS(v) which is isomorphic to its converse is said to be *self-converse*. Kang, Chang, and Yang [3] have shown that a self-converse DTS(v) exists if and only if $v \equiv 0$ or 1 (mod 3) and $v \neq 6$. An *automorphism* of (D, β) is a permutation of D which maps β to itself. An *antiautomorphism* of (D, β) is a permutation of D which maps β to β^{-1}. Clearly, a

DTS(v) is self-converse if and only if it admits an antiautomorphism. Let (S, β') be an STS(v). Let $\beta = \{(a,b,c),(c,b,a) | \{a,b,c\} \in \beta'\}$. Then (S, β) is called the *corresponding* DTS(v), and the identity map on the point set is an antiautomorphism. This yields a self-converse DTS(v) for $v \equiv 1$ or 3 (mod 6).

An antiautomorphism, α, on a DTS(v), (D, β), is called *cyclic* if the permutation defined by α consists of a single cycle of length d and $v-d$ fixed points. Necessary and sufficient conditions for the existence of a DTS(v) admitting a cyclic antiautomorphism have been given by Carnes, Dye, and Reed [1]. We call an antiautomorphism α on a DTS(v), (D, β), *bicyclic* if the permutation defined by α consists of two cycles each of length $N = (v-f)/2$ and f fixed points. In this paper we consider bicyclic antiautomorphisms of directed triple systems with 0 or 1 fixed points.

If N is the length of a cycle, we let the cycles be $(0_i, 1_i, 2_i, \ldots, (N-1)_i)$, $i \in \{0,1\}$, and let ∞ be the fixed point for the 1 fixed point case. Let $\Delta = \{0, 1, 2, \ldots, (N-1)\}$. We shall use all additions modulo N in the triples. For $a_i, b_j, c_k \in D - \{\infty\}$, $i, j, k \in \{0,1\}$, $(a_i, b_j, c_k) \in \beta$, let *orbit* $(a_i, b_j, c_k) = \{((a+t)_i, (b+t)_j, (c+t)_k) | t \in \Delta, t \text{ even}\} \cup \{((c+t)_k, (b+t)_j, (a+t)_i) | t \in \Delta, t \text{ odd}\}$. If ∞ is a fixed point and $a_i, b_j \in \Delta - \{\infty\}$, $i, j \in \{0,1\}$, $(a_i, \infty, b_j) \in \beta$, let *orbit* $(a_i, \infty, b_j) = \{((a+t)_i, \infty, (b+t)_j) | t \in \Delta, t \text{ even}\} \cup \{((b+t)_j, \infty, (a+t)_i) | t \in \Delta, t \text{ odd}\}$. Clearly the orbits partition β.

LEMMA 1: *A* DTS(v) *admitting a bicyclic antiautomorphism with* 0 *or* 1 *fixed points, where $v = 2N$ or $v = 2N+1$, N being the length of each of the cycles, has no orbits of length less than N, except possibly orbits of length $N/2$. If an orbit of length $N/2$ occurs it is only in the* 1 *fixed point case with* $N \equiv 2$ (mod 4).

Proof: Suppose that a DTS(v) exists with a bicyclic antiautomorphism α having an orbit of length $l < N$. Let (a,b,c) be a triple of the short orbit with b not a fixed point. Then $\alpha^l(a,b,c) = (a,b,c)$, thus $\alpha^l(b) = b$, a contradiction to $l < N$. If ∞ is a fixed point, clearly the only orbit of length $l < N$ is orbit $(0_i, \infty, (N/2)_i)$, $i \in \{0,1\}$, for $N \equiv 2$ (mod 4). □

We say that a collection of triples, $\bar{\beta}$, is a collection of *base triples* of a DTS(v) under α if the orbits of the triples of $\bar{\beta}$ produce β and exactly one triple of each orbit occurs in $\bar{\beta}$. Also, we say that the *reverse* of the transitive triple (a,b,c) is the transitive triple (c,b,a).

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LEMMA 2: *Let (D, β) be a* DTS(v) *admitting a bicyclic antiautomorphism with* 0 *fixed points, where $v = 2N$, N being the length of each of the cycles. Then* $v \equiv 16$ (mod 24).

Proof: Suppose N is odd. Let $a, b, c \in D$. Then $\alpha^N(a,b,c) = (c,b,a)$ and we have an STS(v) so that $v \equiv 1$ or 3 (mod 6), which implies that v is odd. Hence N is even.

So $v \equiv 0$ (mod 4). Let $(0_i, 1_i, 2_i, \ldots, (N-1)_i)$, $i \in \{0,1\}$, be one of the cycles.

$(0_i,(N/2)_i)$ occurs in a triple, say $(0_i,(N/2)_i,a_j)$, $(0_i,a_j,(N/2)_i)$, or $(a_j,0_i,(N/2)_i)$, $j\in\{0,1\}$. If $N/2$ is odd then $\alpha^{N/2}(0_i,(N/2)_i,a_j) = ((a+N/2)_j,0_i,(N/2)_i)$, $\alpha^{N/2}(0_i,a_j,(N/2)_i) = (0_i,(a+N/2)_j,(N/2)_i)$, or $\alpha^{N/2}(a_j,0_i,(N/2)_i) = (0_i,(N/2)_i,(a+N/2)_j)$ which leads to the contradiction that the ordered pair $(0_i,(N/2)_i)$ is contained in two distinct triples. Therefore $N/2$ is even so that $N\equiv 0 \pmod 4$, which implies that $v\equiv 0 \pmod 8$. The facts that $v\equiv 0$ or $1 \pmod 3$ and that $v\equiv 0 \pmod 8$ together imply that $v\equiv 0$ or $16 \pmod{24}$.

If $v\equiv 0 \pmod{24}$, the number of triples will be $[24k(24k-1)]/3 = 8k(24k-1)$, and the number of orbits will be $[8k(24k-1)]/12k = 16k-2/3$. Because this quantity cannot be an integer, we must have a short orbit, a contradiction to Lemma 1.

Therefore, we must have $v\equiv 16 \pmod{24}$. □

LEMMA 3: *If* $v\equiv 16 \pmod{24}$ *there exists a* DTS(v) *which admits a bicyclic antiautomorphism with* 0 *fixed points.*

Proof: Let $v=24k+16$, $N=12k+8$.

For $k=0$ the base triples are $(0_0,0_1,4_1)$ and $(0_1,0_0,4_0)$, along with the following and their reverses:
$(0_0,1_1,2_1),(0_0,3_1,6_1),(0_0,5_1,7_1),(0_0,1_0,3_0)$.

For $k=1$ the base triples are $(0_0,0_1,10_1)$ and $(0_1,0_0,10_0)$, along with the following and their reverses:
$(0_0,2_1,8_1),(0_0,3_1,7_1),(0_0,4_1,6_1),(0_0,12_1,19_1),(0_0,13_1,18_1),(0_0,14_1,17_1)$,
$(0_1,4_0,5_0),(0_1,9_0,15_0),(0_1,11_0,19_0),(0_0,2_0,5_0),(0_0,4_0,11_0),(0_1,1_1,9_1)$.

For $k=2$ the base triples are $(0_0,0_1,16_1)$ and $(0_1,0_0,16_0)$, along with the following and their reverses:
$(0_0,2_1,14_1),(0_0,3_1,13_1),(0_0,4_1,12_1),(0_0,5_1,11_1),(0_0,6_1,10_1),(0_0,7_1,9_1)$,
$(0_0,18_1,31_1),(0_0,19_1,30_1),(0_0,20_1,29_1),(0_0,21_1,28_1),(0_0,22_1,27_1)$,
$(0_0,23_1,26_1),(0_1,7_0,8_0),(0_1,15_0,24_0),(0_1,17_0,31_0)$,
$(0_0,2_0,15_0),(0_0,3_0,11_0),(0_0,4_0,10_0),(0_0,5_0,12_0),(0_1,1_1,15_1)$.

For $k\geq 3$ the base triples are $(0_0,0_1,(6k+4)_1)$ and $(0_1,0_0,(6k+4)_0)$, along with the following and their reverses:
$(0_0,2_1,(6k+2)_1),(0_0,3_1,(6k+1)_1),\ldots,(0_0,(3k+1)_1,(3k+3)_1)$,
$(0_0,(6k+6)_1,(12k+7)_1),(0_0,(6k+7)_1,(12k+6)_1),\ldots,(0_0,(9k+5)_1,(9k+8)_1)$,
$(0_1,(3k+1)_0,(3k+2)_0),(0_1,(6k+3)_0,(9k+6)_0),(0_1,(6k+5)_0,(12k+7)_0)$,
$(0_0,(2k+3)_0,(4k+1)_0),(0_0,(2k+4)_0,(4k)_0),\ldots,(0_0,(3k)_0,(3k+4)_0)$,
$(0_0,(4k+4)_0,(6k-1)_0),(0_0,(4k+5)_0,(6k-2)_0),\ldots,(0_0,(5k)_0,(5k+3)_0)$,
$(0_0,2_0,(6k+3)_0),(0_0,(2k-3)_0,(6k)_0),(0_0,(2k-1)_0,(5k+1)_0)$,
$(0_0,(2k)_0,(4k+2)_0),(0_0,(2k+1)_0,(5k+2)_0),(0_1,1_1,(6k+3)_1)$. □

By the previous two lemmas we have the following theorem.

THEOREM 1: *There exists a* DTS(v) *admitting a bicyclic antiautomorphism with* 0 *fixed points if and only if* $v\equiv 16 \pmod{24}$.

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LEMMA 4: *Let (D, β) be a* DTS(v) *admitting a bicyclic antiautomorphism with* 1 *fixed point, where $v = 2N+1$, N being the length of each of the cycles. Then $v \equiv 3 \pmod 6$, with $v \neq 3$.*

Proof. Clearly, since $v = 2N+1$, v must be odd. Also, we must have $v \equiv 0$ or $1 \pmod 3$. Thus $v \equiv 1$ or $3 \pmod 6$.

If $v \equiv 1 \pmod 6$, the number of triples will be $[(6k+1)6k]/3 = 2k(6k+1)$, and the number of orbits will be $[2k(6k+1)]/3k = 4k + 2/3$. Because this quantity cannot be an integer, we must have a short orbit of length $l \neq N/2$, a contradiction to Lemma 1.

Clearly, if $v = 3$, we have three fixed points; therefore, we must have $v \equiv 3 \pmod 6$, with $v \neq 3$. □

LEMMA 5: *If $v \equiv 3 \pmod{12}$, $v \neq 3$, there exists a* DTS(v) *which admits a bicyclic antiautomorphism with* 1 *fixed point.*

Proof. Let $v = 12k+3$, $N = 6k+1$, $k \geq 1$.

Peltesohn [5] proved that a cyclic STS(v) exists if and only if $v \equiv 1$ or $3 \pmod 6$ with $v \neq 9$. Let (S, β) be a cyclic STS(N) with its cyclic automorphism $(0_0, 1_0, 2_0, \ldots, (6k)_0)$. Let β' be a set of base blocks of (S, β).

The base triples are the following: $(0_0, \infty, 0_1)$,
$(0_0, 1_1, (6k)_1), (0_0, 2_1, (6k-1)_1), \ldots, (0_0, (3k)_1, (3k+1)_1)$,
and all triples (a, b, c) where $\{a, b, c\} \in \beta'$. □

LEMMA 6: *If $v \equiv 9 \pmod{24}$ there exists a* DTS(v) *which admits a bicyclic antiautomorphism with* 1 *fixed point.*

Proof. Let $v = 24k+9$, $N = 12k+4$.

For $k = 0$ the base triples are the following and their reverses:
$(0_0, \infty, 2_0), (0_1, \infty, 2_1), (0_0, 0_1, 1_1), (0_1, 1_0, 2_0)$.

For $k \geq 1$ the base triples are the following and their reverses:
$(0_0, \infty, (6k+2)_0), (0_1, \infty, (6k+2)_1)$,
$(0_0, 0_1, (6k+1)_1), (0_0, 1_1, (6k)_1), \ldots, (0_0, (3k)_1, (3k+1)_1)$,
$(0_0, (6k+2)_1, (12k+2)_1), (0_0, (6k+3)_1, (12k+1)_1), \ldots, (0_0, (9k+1)_1, (9k+3)_1)$,
$(0_1, (3k+2)_0, 1_0)$,
$(0_0, (2k+1)_0, (4k+1)_0), (0_0, (2k+2)_0, (4k)_0), \ldots, (0_0, (3k)_0, (3k+2)_0)$,
$(0_0, (4k+2)_0, (6k+1)_0), (0_0, (4k+3)_0, (6k)_0), \ldots, (0_0, (5k+1)_0, (5k+2)_0)$. □

LEMMA 7: *If $v \equiv 21 \pmod{24}$ there exists a* DTS(v) *which admits a bicyclic antiautomorphism with* 1 *fixed point.*

Proof. Let $v = 24k+21$, $N = 12k+10$.

For $k = 0$ the base triples are $(0_0, 0_1, 4_0)$, $(0_0, 5_1, 6_0)$, $(0_1, 0_0, 4_1)$, $(0_1, 1_0, 8_1)$,

$(0_1, 2_0, 3_1)$, $(0_1, 3_0, 6_1)$, $(0_1, 6_0, 2_1)$, $(0_1, 8_0, 1_1)$, $(1_1, 0_0, 2_1)$ and $(1_1, 6_0, 4_1)$, along with the following and their reverses:
$(0_0, \infty, 5_0), (0_1, \infty, 5_1), (0_0, 1_0, 3_0)$.

For $k = 1$ the base triples are $(0_0, 4_0, 12_0)$, $(0_0, 5_0, 10_0)$, $(0_0, 7_0, 14_0)$ and $(0_0, 9_0, 18_0)$, along with the following and their reverses:
$(0_0, \infty, 11_0), (0_1, \infty, 11_1)$,
$(0_0, 0_1, 10_1), (0_0, 1_1, 9_1), \ldots, (0_0, 4_1, 6_1)$,
$(0_0, 12_1, 21_1), (0_0, 13_1, 20_1), \ldots, (0_0, 16_1, 17_1)$,
$(0_1, 11_0, 17_0), (0_0, 1_0, 3_0)$.

For $k = 2$ the base triples are $(0_0, 4_0, 12_0)$, $(0_0, 11_0, 22_0)$, $(0_0, 13_0, 26_0)$ and $(0_0, 15_0, 30_0)$, along with the following and their reverses:
$(0_0, \infty, 17_0), (0_1, \infty, 17_1)$,
$(0_0, 0_1, 16_1), (0_0, 1_1, 15_1), \ldots, (0_0, 7_1, 9_1)$,
$(0_0, 18_1, 33_1), (0_0, 19_1, 32_1), \ldots, (0_0, 25_1, 26_1)$,
$(0_1, 17_0, 26_0), (0_0, 1_0, 6_0), (0_0, 2_0, 16_0), (0_0, 3_0, 10_0)$.

For $k = 3$ the base triples are $(0_0, 4_0, 20_0)$, $(0_0, 13_0, 26_0)$, $(0_0, 15_0, 30_0)$ and $(0_0, 21_0, 42_0)$, along with the following and their reverses:
$(0_0, \infty, 23_0), (0_1, \infty, 23_1)$,
$(0_0, 0_1, 22_1), (0_0, 1_1, 21_1), \ldots, (0_0, 10_1, 12_1)$,
$(0_0, 24_1, 45_1), (0_0, 25_1, 44_1), \ldots, (0_0, 34_1, 35_1)$,
$(0_1, 23_0, 35_0), (0_0, 1_0, 19_0), (0_0, 2_0, 11_0), (0_0, 3_0, 10_0), (0_0, 5_0, 22_0), (0_0, 6_0, 14_0)$.

For $k = 4$ the base triples are $(0_0, 4_0, 12_0)$, $(0_0, 23_0, 46_0)$, $(0_0, 25_0, 50_0)$ and $(0_0, 27_0, 54_0)$, along with the following and their reverses:
$(0_0, \infty, 29_0), (0_1, \infty, 29_1)$,
$(0_0, 0_1, 28_1), (0_0, 1_1, 27_1), \ldots, (0_0, 13_1, 15_1)$,
$(0_0, 30_1, 57_1), (0_0, 31_1, 56_1), \ldots, (0_0, 43_1, 44_1)$,
$(0_1, 29_0, 44_0), (0_0, 1_0, 14_0), (0_0, 2_0, 24_0), (0_0, 3_0, 21_0)$,
$(0_0, 5_0, 16_0), (0_0, 6_0, 26_0), (0_0, 7_0, 17_0), (0_0, 9_0, 28_0)$.

For $k = 5$ the base triples are $(0_0, 4_0, 12_0)$, $(0_0, 29_0, 58_0)$, $(0_0, 31_0, 62_0)$ and $(0_0, 33_0, 66_0)$, along with the following and their reverses:
$(0_0, \infty, 35_0), (0_1, \infty, 35_1)$,
$(0_0, 0_1, 34_1), (0_0, 1_1, 33_1), \ldots, (0_0, 16_1, 18_1)$,
$(0_0, 36_1, 69_1), (0_0, 37_1, 68_1), \ldots, (0_0, 52_1, 53_1)$,
$(0_1, 35_0, 53_0), (0_0, 1_0, 23_0), (0_0, 2_0, 30_0), (0_0, 3_0, 19_0), (0_0, 5_0, 20_0)$,
$(0_0, 6_0, 32_0), (0_0, 7_0, 21_0), (0_0, 9_0, 34_0), (0_0, 10_0, 27_0), (0_0, 11_0, 24_0)$.

For $k = 6$ the base triples are $(0_0, 1_0, 2_0)$, $(0_0, 6_0, 4_0)$, $(0_0, 39_0, 78_0)$ and $(0_0, 79_0, 76_0)$, along with the following and their reverses:
$(0_0, \infty, 41_0), (0_1, \infty, 41_1)$,
$(0_0, 0_1, 40_1), (0_0, 1_1, 39_1), \ldots, (0_0, 19_1, 21_1)$,
$(0_0, 42_1, 81_1), (0_0, 43_1, 80_1), \ldots, (0_0, 61_1, 62_1)$,
$(0_1, 41_0, 62_0), (0_0, 5_0, 34_0), (0_0, 7_0, 30_0), (0_0, 8_0, 36_0), (0_0, 9_0, 26_0)$,
$(0_0, 10_0, 32_0), (0_0, 11_0, 31_0), (0_0, 12_0, 37_0), (0_0, 13_0, 40_0), (0_0, 14_0, 38_0)$,
$(0_0, 15_0, 33_0), (0_0, 16_0, 35_0)$.

For $k \geq 7$ the base triples are $(0_0, 1_0, 2_0)$, $(0_0, 3_0, 6_0)$, $(0_0, 4_0, (12k + 8)_0)$ and $(0_0, (6k + 3)_0, (12k + 6)_0)$, along with the following and their reverses:

$(0_0, \infty, (6k+5)_0), (0_1, \infty, (6k+5)_1),$
$(0_0, 0_1, (6k+4)_1), (0_0, 1_1, (6k+3)_1), \ldots, (0_0, (3k+1)_1, (3k+3)_1),$
$(0_0, (6k+6)_1, (12k+9)_1), (0_0, (6k+7)_1, (12k+8)_1), \ldots, (0_0, (9k+7)_1, (9k+8)_1),$
$(0_1, (6k+5)_0, (9k+8)_0),$
$(0_0, (2k+6)_0, (4k-1)_0), (0_0, (2k+7)_0, (4k-2)_0), \ldots, (0_0, (3k-1)_0, (3k+6)_0),$
$(0_0, (4k+4)_0, (6k)_0), (0_0, (4k+5)_0, (6k-1)_0), \ldots, (0_0, (5k-2)_0, (5k+6)_0),$
$(0_0, 5_0, (5k+4)_0), (0_0, (2k-5)_0, (5k)_0), (0_0, (2k-3)_0, (4k+2)_0),$
$(0_0, (2k-2)_0, (5k+2)_0), (0_0, (2k-1)_0, (5k+1)_0), (0_0, (2k)_0, (6k+1)_0),$
$(0_0, (2k+1)_0, (6k+4)_0), (0_0, (2k+2)_0, (6k+2)_0),$
$(0_0, (2k+3)_0, (5k+3)_0), (0_0, (2k+4)_0, (5k+5)_0).$ □

By the previous four lemmas we have the following theorem.

THEOREM 2: *There exists a* DTS(v) *admitting a bicyclic antiautomorphism with* 1 *fixed point if and only if* $v \equiv 3 \pmod 6$, *with* $v \neq 3$.

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