More on the Linear k-arboricity of Regular Graphs

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Abstract

Bermond et al. [5] conjectured that the edge set of a cubic graph G can be partitioned into two *linear k-forests*, that is to say two forests whose connected components are paths of length at most k, for all $k \geq 5$. That the statement is valid for all $k \geq 18$ was shown in [8] by Jackson and Wormald. Here we improve this bound to

$$k \geq \begin{cases} 7 & \text{if } \chi'(G) = 3; \\ 9 & \text{otherwise.} \end{cases}$$

The result is also extended to d-regular graphs for $d > 3$, at the expense of increasing the number of forests to $d - 1$.

All graphs considered will be finite. We shall refer to graphs which may contain loops or multiple edges as *multigraphs* and reserve the term *graph* for those which do not. A *linear forest* is a forest each of whose components is a path. The *linear arboricity* of a graph G, defined by Harary [7], is the minimum number of linear forests required to partition $E(G)$ and is denoted by $la(G)$. It was shown by Akiyama, Exoo and Harary [1] that $la(G) = 2$ when G is cubic. A *linear k-forest* is a forest consisting of paths of length at most k. The *linear k-arboricity* of G, introduced by Bermond et al. [5], is the minimum number of linear k-forests required to partition $E(G)$, and is denoted by $la_k(G)$. When such a partition of $E(G)$ has been imposed, we say that G has been *factored* into linear k-forests. We refer to each linear forest in such a partition of $E(G)$ as a *factor* of G and the partition itself is called a *factorization.*

It is conjectured in [5] that if G is cubic then $la_5(G) \leq 2$. A partial result is obtained by Delamarre et al. [6] who show that $la_k(G) \leq 2$ when $k \geq \frac{1}{2}|V(G)| \geq 4$.

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Jackson and Wormald [8] improved this result when $|V(G)| \geq 36$, showing that, for $k \geq 18$, an integer, $la_k(G) = 2$. Here we shall improve on this further to show the following.

Theorem 1. *Let G be a cubic graph and let k be an integer. Then $la_k(G) = 2$ for all*

$$k \geq \begin{cases} 7 & \text{if } \chi'(G) = 3; \\ 9 & \text{otherwise.} \end{cases}$$

In [9], Lindquester and Wormald considered the following variation of linear arboricity for r-regular graphs. An r-regular graph G is said to be (l, k)*-linear arborific* if it can be factored into l linear k-forests. In this setting, we are able to prove the following.

Theorem 2. *Let G be an r-regular graph, $r \geq 3$, and let k be an integer. Then G is $(r-1, k)$-linear arborific for all*

$$k \geq \begin{cases} 7 & \text{if } \chi'(G) = r; \\ 9 & \text{otherwise.} \end{cases}$$

While Theorems 1 and 2 are proved here for finite graphs, the results also apply for infinite graphs using a standard method (by Tihonov's theorem – see for example the application on page 57 of [2]).

Before we prove the theorems, we shall present some preliminary results which indicate some restrictions we can demand of factorizations of cubic graphs. These will be most useful in the proof of the theorem.

For our first result we introduce the following terminology. An *odd linear forest* is a linear forest in which each component is a path of odd length. Also, we use $\chi'(G)$ to denote the chromatic index of G (i.e. the minimum number of colours required to colour the edges of G so that no two edges of the same colour are incident with the same vertex).

Lemma. *Let G be a cubic graph. Then G can be factored into two odd linear forests if and only if $\chi'(G) = 3$.*

Proof. Suppose first that G is a cubic graph with a factorization, (F_1, F_2) into odd linear forests F_1 and F_2. Colour the edges of the paths in F_1 alternately red and blue so that each path in F_1 has its first and last edges coloured red. Similarly, colour the edges of the paths in F_2 alternately green and blue so that each path in F_2 has its first and last edges coloured green. This yields a proper 3-edge-colouring of G giving $\chi'(G) = 3$.

Conversely, let us suppose that $\chi'(G) = 3$ and that we have a proper 3-edge-colouring imposed on G using the colours blue, green and red. Let F_1' and F_2' be factors of G induced by the blue and green edges and by the red edges respectively. Thus F_1' consists of disjoint even cycles, while F_2' is a set of disjoint edges covering the vertices of G. Form new factors F_1 and F_2 where F_2 is the subgraph of G induced by the edges in F_2' together with at most one edge from each cycle in F_1' chosen so that F_2 is acyclic and such that the paths in F_2 have maximum possible total length. The factor F_1 consists of F_1' with the edges in F_2 removed.

Claim: F_2 contains an edge from each cycle in F_1'. To see this, assume to the contrary that there is a cycle $C = v_1v_2\dots v_kv_1$ in F_1. Then v_1 and v_2 are both ends of the one path in F_2, while v_2 and v_3 are both ends of the one path in F_2 (by the maximality of total length of paths in F_2). Since G is a graph, $v_1 \neq v_3$ and no path has three ends so the claim follows.

Thus each path in F_1 has odd length and each path in F_2 has odd length (these paths must begin and end with red edges and have every second edge red throughout their lengths). ∎

Corollary. *Let G be an r-regular graph. If $\chi'(G) = r$, then G can be factored into $r-1$ odd linear forests.*

Proof. Since $\chi'(G) = r$, the edges of G can be partitioned into r 1-factors. choose $r-3$ of these to form factors in our factorization, the remaining 3 1-factors inducing a cubic spanning subgraph of G which has chromatic index 3. By the lemma, this subgraph can be factored into two odd linear forests, completing the desired factorization and the proof of the corollary. ∎

Proposition. *Let G be a cubic graph. Then there is a factorization of G into two linear forests, F_1 and F_2 such that no component in F_i isomorphic to P_3 has its end vertices adjacent in G via an internal edge of a path in F_j, where $\{i, j\} = \{1, 2\}$. (Such a factorization is to have the spread-P_3 property.)*

Proof. The proposition is certainly true for $G = K_4$. So assume that G is a cubic graph of smallest order which does not admit a factorization with the spread-P_3 property. Clearly, G cannot be triangle-free. But if we have a triangle T, with each edge of T belonging to precisely one triangle, then contracting T to a vertex yields a cubic graph, G', of smaller order than than G. By the minimality of G, G' admits a factorization with the spread-P_3 property. This factorization easily lifts to a factorization of G with the spread-P_3 property.

Consequently, we may assume that all triangles in G are paired. By [1] we can let G be factored into two linear forests. Assume this is done so that there are as few unspread-P_3's as possible. (By *unspread-P_3* we mean a path of length two in one factor whose endvertices are adjacent in G via an internal edge in the other factor.) As G was chosen not to admit a factorization with the spread-P_3 property, we must have an unspread-P_3 in one of the paired triangles. Figure 1 indicates how this factorization may be altered to reduce the number of unspread-P_3's. This contradiction establishes the result. ∎

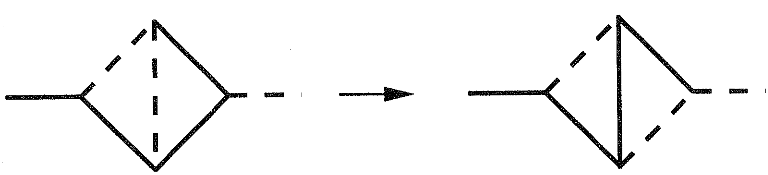

Figure 1.

Proof of Theorem 1. Define $\mathcal{F} = \mathcal{F}(G)$ to be the set of (F_1, F_2) such that F_1 and F_2 form a factorization of G with the spread-P_3 property. Then $\mathcal{F}$ is nonempty by the proposition. For $(F_1, F_2) \in \mathcal{F}$, let $l(F_1, F_2)$ denote the length of the longest path in F_1 or F_2. Let t be the minimum number such that there exists $(F_1, F_2) \in \mathcal{F}$ with $l(F_1, F_2) = t$.

Choose $(F_1, F_2) \in \mathcal{F}$ with $l(F_1, F_2) = t$ and such that, subject to this condition, the total number $m(F_1, F_2)$, of paths of length t in F_1 and F_2 is as small as possible. By symmetry, we may assume that F_1 has a path P of length t.

Suppose, contrary to Theorem 1, that $t \geq 10$. We will use the structure of this factorization to produce distinct edges, e_i of G for each integer $i \geq 0$, thereby contradicting the finiteness of G. To assist in this process we define an i-improvement.

Given path sets $S_0, \ldots, S_i$, define $R_i = R_i(S_0, \ldots, S_i)$ to be the set of all edges of G which are either contained in or incident with any of the paths in $S_0, \ldots, S_i$.

Given $(F_1, F_2) \in \mathcal{F}$ and path sets $S_0, \ldots, S_i$, we say that $(F_1', F_2') \in \mathcal{F}$ is an *i-improvement* of (F_1, F_2) if there exists an edge e of a path in S_i such that

(i) all edges in $R_i \setminus e$ are in F_1' iff they are in F_1,

(ii) e is in F_1' iff it is not in F_1, and

(iii) each path in F_j' of length at least t is also a path in F_j, $j = 1$ and 2.

The algorithm below selects paths from the factors F_1 and F_2 by first taking an edge e from a special set of edges in the paths already chosen. The edges in this set are specifically determined so that the endvertices of each one are also the end vertices of paths (in one of the factors F_1 or F_2) not yet selected in the algorithm. After e is selected, the paths which emanate from its endvertices in turn contain new edges which go into the special set of edges, thus providing more edges with which to continue the process. We refer to these special edges as *live* edges as, at each iteration, they indicate paths which may be chosen to extend our set of selected paths. The endvertex of a newly selected path not incident with e may also be incident with another live edge, f. Such an edge f no longer indicates two unchosen paths, and thus we do not consider it to be live any more. We say that such an edge f has been *killed* by the end vertices of the chosen path.

We show that the number of live edges does not decrease as the algorithm proceeds if t, the length of a longest path in the factorization, is at least 10. Consequently, our set of chosen paths grows without bound contradicting the finiteness of G.

Initially we set:

$S_0 := \{P\}$ (our first chosen path);

$T_0 :=$ a maximum set of independent internal edges in P excluding those third from either end (live edges which are available to use in Step 2 to determine new paths with which to extend our chosen set);

$W_0 := \{$end vertices of $P\}$ (these vertices may kill live edges at some later stage).

Note that when $t \geq 5$, $|T_0| = \lceil \frac{t-2}{2} \rceil$. So for $t \geq 7$ we have have $|T_0| > |W_0|$.

For $i > 0$ carry out the following three steps.

1. Choose $e_{i-1} \in T_{i-1}$.
2. Repeat (a) and (b) consecutively until (b) requires no action because its condition fails:
   (a) Define S_i to be the set of paths in the factorization (F_1, F_2) which terminate at the endvertices of e_{i-1}.(If e_{i-1}belongs to a path in $F_1(F_2)$, then $S_i \subseteq F_2(F_1)$.)
   (b) If there is an i-improvement (F_1', F_2') of (F_1, F_2) then reset (F_1, F_2) to be equal to (F_1', F_2').
3. Determine the sets T_i and W_i by the following method. Set:

   $V_i :=$ {endvertices of paths in S_i}\{endvertices of e_{i-1}};

   $N_i :=$ a maximum set of independent internal edges of paths in S_i other than those third along from a vertex in V_i ;

   $T_i' := (T_{i-1} \setminus \{e_{i-1}\}) \cup N_i$ (adding edges in N_i to the set of live edges, and deleting e_{i-1}, which has just been killed);

   $W_i' := W_{i-1} \cup V_i$ (V_i contains new vertices with the potential to kill live edges);

   $T_i := T_i' \setminus \{e \in T_i'$ incident with a vertex in $W_i'\}$ (these edges have been killed);

   $W_i := W_i' \setminus \{w \in W_i'$ incident with and edge in $T_i'\}$ (after a vertex kills a live edge, it no longer has the potential to kill further live edges).

Step 2 must eventually terminate because the redefinition of (F_1, F_2) in 2(b) leads to a strictly smaller total length of paths in the set S_i when 2(a) is next performed.

Denote the value of (F_1, F_2) at the end of the ith iteration of the two steps above by $(F_{i,1}, F_{i,2})$. Then for any $j > i$ the set S_i retains its property of being a set of maximal paths in $F_{j,1}$ or $F_{j,2}$, because the redefinition of (F_1, F_2) does not disturb any edges in R_{j-1}. Thus, $(F_{j,1}, F_{j,2})$ cannot have any i-improvement for $i < j$.

Note also that live edges in N_i are killed by vertices in W_i' precisely when such an edge shares an endvertex with a path in some $S_j, j \leq i-1$. In this way, the paths in S_i cannot be included in $S_j, j \leq i-1$. Thus an edge $e_h \in T_h$ chosen in Step 1 cannot be contained in $T_{h'}$ whenever $h' > h$.

Consequently, having chosen $e_{i-1} \in T_{i-1}$ for Step 1, we have one of two possiblities at the completion of Step 2.

(i) There is a single path, P', in S_i joining the endvertices of e_{i-1}. Since we are working with factorizations with the spread-P_3 property, P' must have length at least 3. As we work through Step 3 in this case, $V_i = \emptyset$ and we have at least one edge in N_i. Each edge of T_i' killed by a vertex in W_i' results in one vertex from W_{i-1} not being retained in W_i and, in this case, no new vertices will be added to W_{i-1} when forming W_i (since $V_i = \emptyset$). Thus $|T_i|$ increases by at least as much as $|W_i|$.

(ii) There are two paths $\tilde{P}, \hat{P} \in S_i$ terminating at the endvertices of e_{i-1}. Suppose that $l(\tilde{P}) + l(\hat{P}) \leq t-2$. Then e_{i-1} cannot be in $S_0 = P$ (if it were, then swapping e_{i-1} between the appropriate factors would produce a factorization in $\mathcal{F}$ in which the total number of paths of length at least t is less than in our originally chosen factorization; a contradiction). Thus e_{i-1} is in some path in $S_j, 1 \leq j < i$. But now swapping e_{i-1} between the appropriate factors will yield a j-improvement for some $j < i$. (The internal edges third from an endvertex of a path in $S_j, 0 \leq j \leq i-1$ have been excluded from T_{i-1} to ensure that swapping e_{i-1} between factors cannot result in an unspread-P_3, producing a factorization not in $\mathcal{F}$.). Thus we have $l(\tilde{P})+l(\hat{P}) \geq t-1$. As we have assumed that $t \geq 10$, this means that $l(\tilde{P}) + l(\hat{P}) \geq 9$.

Working through Step 3 in this case, we look at the combined contribution to N_i from both $\tilde{P}$ and $\hat{P}$. Now consider $\pi \in \{\tilde{P}, \hat{P}\}$. If $l(\pi) \in \{3,4\}$, π contributes one edge to N_i. If $l(\pi) \in \{5,6\}$, π contributes two edges to N_i. If $l(\pi) \geq 7$, π contributes at least three edges to N_i. Consequently, given that $l(\tilde{P}) + l(\hat{P}) \geq 9$, at least three edges are added to N_i. (Note that if we restrict our paths in F_1 and F_2 to be of odd length, then the above considerations lead to at least three edges being contributed to N_i when $l(\tilde{P})+l(\hat{P}) \geq 8$, i.e. when $t \geq 9$.) As before, each edge of T_i' killed by a vertex in W_i' results in one vertex from W_i' not being retained in W_i. There are at most two new vertices added to W_{i-1} to form W_i' and at least three new edges added to T_{i-1} to obtain T_i'. Once again, $|T_i|$ increases by at least as much as $|W_i|$.

In both cases we see by induction that $|T_i|$ (the number of live edges) exceeds $|W_i|$ (the number of vertices each with the potential to kill a live edge). Thus we may continue indefinitely, removing a different edge from T_i at each iteration. This contradicts the finiteness of G.

Next assume that $\chi'(G) = 3$. By the Lemma, G can be factored into two odd linear forests. Working through the proof above restricting $\mathcal{F}$ to the set of factorizations of G into two odd linear forests, we make the following minor modifications.

$T_0 :=$ every second edge in P;

$N_i :=$ every second edge in paths in S_i.

Now, everything follows as before since in case (ii) above, the paths $\tilde{P}, \hat{P}$ are restricted to be of odd length. As noted in the parenthetic comment in case (ii), the finiteness of G is contradicted if $t \geq 9$ when all paths are of odd length. We conclude that the longest (odd) path required in our factorization has length at most 7 and the result follows. ∎

Finally we note that our Theorem can easily be extended to graphs of maximum degree three since any such graph can be embedded as a subgraph of a cubic graph. Using this we deduce the following:

Corollary. *Let G be a 4-regular graph and $k \geq 9$ be an integer. Then $la_k(G) = 3$.*

Proof. Using [10] we may choose a 2-factor F in G. Clearly F has a spanning k-linear forest D. Since $H = G - E(D)$ has maximum degree three, it follows from the above-mentioned extension of our Theorem that $la_k(H) = 2$. Thus $la_k(G) = 3$. ∎

Having established Theorem 1 and its corollary above, we may now prove Theorem 2.

Proof of Theorem 2. Let G be an r-regular graph with $r \geq 3$. If $\chi'(G) = r$, then we may factor G into r 1-factors. Choose $r - 3$ of these 1-factors and delete them leaving a cubic spanning subgraph of G which is 3-edge-colourable. By our theorem, this subgraph is $(2, 7)$-linear arborific and thus G is $(r - 1, 7)$-linear arborific as required.

When $\chi'(G) \neq r$, we proceed by induction on r. Our theorem and the above corollary show that this is true for $r = 3, 4$. Assume that the statement is true for all $r \leq r_0$ and let G be an $r_0 + 1$-regular graph. If $r_0 + 1$ is even, then G has a 2-factor, F, say. The graph $G - F$ is regular of degree $r_0 - 1$ and so, by induction, may be factored into $r_0 - 2$ linear 9-forests. The desired factorization is completed by decomposing F into two linear 2-forests in the obvious way.

So assume that $r_0 + 1$ is odd. Let M be a maximum matching in G. Then $G - M$ has vertices of degrees r_0 and $r_0 + 1$, and the vertices of degree $r_0 + 1$ form an independent set, B, say. The subgraph of $G - M$ induced by $B \cup N(B)$ with edges between vertices in $N(B)$ deleted is bipartite and, by Hall's theorem and considerations of degrees, contains a matching, M', which covers the vertices in B. Thus $G' = G - (M \cup M')$ is biregular with degrees r_0 and $r_0 - 1$. If H' is an isomorphic copy of G' and G'' is the graph obtained from G' and H' by adding a matching between the vertices of degree $r_0 - 1$ in G' and the vertices of degree $r_0 - 1$ in H', then G'' is r_0-regular and, by our inductive hypothesis, contains a factorization into $r_0 - 1$ linear 9-forests. Restricting these factors to the vertices in G together with $M \cup M'$ gives the desired factorization of G into r_0 linear 9-forests. ($M \cup M'$ is a linear forest with components of size 1 and 2.) ∎

We note that the technique of Alon in [3] will also produce results along these lines, but only for larger d. In fact, for large enough d every d-regular graph can be shown to be $(d - 1, 2)$-linear arborific (Alon [4]).

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