On clique polynomials

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Abstract

Let G be a simple graph. We assign a polynomial $C(G;x)$ to G, called the clique polynomial, where the coefficient of x^i, $i > 0$, is the number of cliques of G with i vertices, and the constant term is 1. Fisher and Solow (1990), proved that this polynomial always has a real root. We prove this result by a simple and elementary method, which also implies the following results. If ζ_G is the greatest real root of $C(G;x)$ then for an induced subgraph H of G, $\zeta_H \leq \zeta_G$, and for a spanning subgarph H of G, $\zeta_H \geq \zeta_G$. As a consequence of the first inequality we have $\alpha(G) \leq -1/\zeta_G$, where $\alpha(G)$ denotes the independence number of G.

1 Introduction

Throughout this paper we consider simple graphs, i.e. finite undirected graphs with no loops and multiple edges, and we use the terminology and notation of [1].

The *dependence polynomial* was first introduced by Fisher [2], who studied the following problem: How many n letter words can be made from an m letter alphabet if certain pairs of letters commute? Fisher and Solow [3] defined the dependence polynomial as follows:

$$f_G(x) = 1 - c_1x + c_2x^2 - c_3x^3 + \cdots + (-1)^\omega c_\omega x^\omega;$$

where ω is the size of the largest clique in G and c_i denotes the number of complete subgraphs of size i in G. For a set S of words with an operation on them we assign a graph G_S such that $V(G_S) = S$ and two vertices are joined iff they commute. Fisher [2] proved that the generating function for the above problem is precisely $\frac{1}{f_{G_S}(x)}$.

If we change the signs of all negative coefficients in $f_G(x)$ to positive signs, we obtain a polynomial which is called the *clique polynomial* of G. Using the notation

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of [4] we denote it by $C(G;x)$. So we have:

$$C(G;x) = 1 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_\omega x^\omega.$$

In [3] Fisher and Solow showed that the dependence polynomial of a graph always has a real root. In fact they prove that the smallest root (in absolute value) of $f_G(x)$ is real. This result immediately implies the existence of a real root for the clique polynomial.

In this paper we give a simple proof of the later result. In addition, we show that there is some relation between the largest negative root of $C(G;x)$ and that of $C(H;x)$, for special subgraphs H of G.

2 Results

We first present the following observation and then use it as the main tool to prove our main theorems.

Lemma 1. *Let G be a graph and let $v \in V(G)$. Then*

(*a*) $C(G;x) = C(G \setminus v;x) + xC(G[N(v)];x)$; *where $N(v)$ is the neighborhood of v.*

(*b*) $C(G;x) = C(G - uv;x) + x^2C(G[N(u) \cap N(v)];x)$; *where $uv \in E(G)$.*

Proof. Let A_i be an i-clique of G. (a) Either $v \notin A_i$, then A_i is an i-clique in $G \setminus v$; or $v \in A_i$, then A_i is obtained from an $(i-1)$-clique of $G[N(v)]$. Summing up the number of these two kinds of i-cliques we obtain relation (a).
(b) Either A_i does not contain the edge uv, then A_i is an i-clique in $G - uv$; or it does contain uv, then A_i is obtained from an $(i-2)$-clique of $G[N(u) \cap N(v)]$. Summing up the number of these two kinds of i-cliques we obtain relation (b). □

To pursue our study we need the following notation:

Notation. *Let G be a graph and let $\mathcal{Z}(G)$ be the set of negative real roots of $C(G;x)$. If $\mathcal{Z}(G)$ is non-empty then define ζ_G to be* $\max \mathcal{Z}(G)$ *and otherwise to be $-\infty$.*

The following theorem plays an essential role where we reprove the result of Fisher and Solow. Also it presents a nice property of ζ_G in conjunction with induced subgraphs.

Theorem 1. *If G is a graph and H is one of its induced subgraph, then $\zeta_H \leq \zeta_G$.*

Proof. Let $n = |V(G)|$. We prove the theorem by induction on n. For $n = 1$ and 2 the assertion is obvious. If H is an arbitrary proper induced subgraph of G, then we can find a vertex v of G such that H is an induced subgraph of $G \setminus v$. Hence it is sufficient to prove the theorem for induced subgraphs of the form $G \setminus v$, for some $v \in V(G)$. Now, let $v \in V(G)$. If $\mathcal{Z}(G \setminus v) = \emptyset$ then there is nothing to prove. So we can assume that $\mathcal{Z}(G \setminus v)$ is not empty. On other hand, by part (a) of Lemma 1 we have:

$$C(G;x) = C(G \setminus v;x) + xC(G[N(v)];x).$$

Substituting $\zeta_{G\setminus v}$ in both sides of the above equation and applying induction we have $C(G[N(v)], \zeta_{G\setminus v}) \geq 0$, thus $C(G, \zeta_{G\setminus v}) \leq 0$. On the other hand $C(G, 0) = 1$. So the theorem is proved. □

Theorem 2. *For every graph G, $-1 \leq \zeta_G < 0$.*

Proof. Let u be a vertex of G, and H be the subgraph induced on u. Clearly $\zeta_H = -1$. Thus by the above theorem we must have $\zeta_G \geq -1$, as desired. □

Turan's theorem for triangle-free graphs is a consequence of Theorem 2 :

Corollary 1. *If G is a triangle-free graph then $|E(G)| \leq |V(G)|^2/4$.*

Proof. Since G has no triangle we have:

$$C(G;x) = 1 + |V(G)|x + |E(G)|x^2.$$

By Theorem 2, $C(G;x)$ has a real root which implies that the discriminant of this polynomial i.e. $|V(G)|^2 - 4|E(G)|$ is non-negative; as claimed. □

The two following propositions are obtained by considering some special induced subgraphs.

Proposition 1. *Let G be a graph and $\alpha(G)$ be the independence number of G. Then*

$$\alpha(G) \leq -1/\zeta_G.$$

Proof. Consider the subgraph H induced by an independent set of size $\alpha(G)$. We have $\zeta_H = -1/(\alpha(G))$ and by Theorem 1, $\zeta_H \leq \zeta_G$. This proves the proposition. □

Proposition 2. *Let G be a graph which is not complete and let $g(G)$ be the girth of G. Then*

$$g(G) \leq \frac{-1}{\zeta_G^2 + \zeta_G}$$

Proof. Consider a cycle of G with the size $g(G)$. This is an induced subgraph of G. Calculating the ζ of this cycle and applying Theorem 1 we obtain the desired inequality. □

Remark 1. By the same method one can prove a similar assertion with $g(G)$ replaced by the length of the the smallest odd cycle.

The following corollary is an immediate consequence of Proposition 1:

Corollary 2. *For every graph G, $\chi(G) \geq -|V(G)|\zeta_G$.*

Theorem 3. *If G is a graph and H is a spanning subgraph of G, then $\zeta_G \leq \zeta_H$.*

Proof. It is enough to prove the theorem in the case of $H = G - e$ where e is an edge of G. Suppose $e = uv$ for $u, v \in V(G)$. By the part (b) of Lemma 1 we have:

$$C(G;x) = C(G - uv; x) + x^2 C(G[N(u) \cap N(v)]; x),$$

where $uv \in E(G)$. Substitute ζ_G in both sides of the above equation. We obtain:

$$C(G - uv; \zeta_G) = -\zeta_G^2\, C(G[N(u) \cap N(v)]; \zeta_G). \tag{1}$$

On the other hand $G[N(u) \cap N(v)]$ is an induced subgraph of G and therefore by Theorem 1 the right hand side of equation (1) is negative, which implies that $C(G - uv; \zeta_G)$ is negative also. This together with the fact that $C(G - uv; 0) = 1$, implies the assertion. □

We can apply Theorem 3 to prove some necessary conditions for existence of Hamiltonian cycles and perfect matchings which are useful in some special cases.

Corollary 3. *Let G be a graph with n vertices. We have:*

(a) If $n \geq 4$ and $\zeta_G > \frac{-1+\sqrt{1-4/n}}{2}$, then G is not Hamiltonian.

(b) If $n \geq 2$ and $\zeta_G > -1 + \sqrt{1 - 2/n}$, then G does not have perfect matching.

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