Packing, Perfect Neighbourhood, Irredundant and R-annihilated Sets in Graphs

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Abstract

A variety of relationships between graph parameters involving packings, perfect neighbourhood, irredundant and R-annihilated sets is obtained. Some of the inequalities are improvements of existing bounds for the lower irredundance number, and others are motivated by the conjecture (recently disproved) that for any graph the smallest cardinality of a perfect neighbourhood set is at most the lower irredundance number.

1. Introduction

This work is concerned with properties of four kinds of vertex subsets X of a simple graph G, namely packing, perfect-neighbourhood, irredundant and R-annihilated sets. The first task is their definition and to observe that each may be characterized in terms of a certain partition of the vertex set V of G induced by X. We denote by $N(X)$ ($N[X]$) the open (closed) neighbourhood of the set X. As usual $N(\{x\})$ and $N[\{x\}]$ will be abbreviated to $N(x)$ and $N[x]$. For $A, B \subseteq V$, we say that A *dominates* B, written $A \succ B$, (or B *is dominated by* A) if $B \subseteq N[A]$. The *private neighbourhood* $pn(x, X)$ of x in X is defined by

$$pn(x, X) = N[x] - N[X - \{x\}].$$

An element u of $pn(x, X)$ is called a *private neighbour* of x relative to X and is one of two types. Either u is an isolate of $G[X]$, in which case $u = x$, or $u \in V - X$

and is adjacent to precisely one vertex of X. The latter type is called an *external private neighbour* (epn) of x. The concept of private neighbourhood enables us to define from X, a partition $\mathcal{P}(X) = Z_X \cup Y_X \cup E_X \cup F_X \cup C_X \cup R_X$ (disjoint union) of V, where:

$$Z_X = \{x \in X \mid x \text{ is isolated in } G[X]\},$$

$$Y_X = X - Z_X,$$

$$E_X = \{v \in V - X \mid v \text{ is an epn of some } y \in Y_X\},$$

$$F_X = \{v \in V - X \mid v \text{ is an epn of some } z \in Z_X\},$$

$$C_X = \{v \in V - X \mid |N(v) \cap X| \geq 2\}$$

$$\text{and } R_X = V - N[X].$$

When the basic subset X is clear from the context, we will omit the subscripts X. Many familiar properties of vertex subsets X may be defined in terms of the partition $\mathcal{P}(X)$. For example, X is independent if $Y = \emptyset$, dominating if $R = \emptyset$ and total dominating if $R \cup Z = \emptyset$. We now define the first three of our principal concepts. The set X is a *packing set* (or simply a *packing*) if $N[x_1] \cap N[x_2] = \emptyset$ for all distinct $x_1, x_2 \in X$, an *irredundant set* if for all $x \in X$, $pn(x, X) \neq \emptyset$ and a *perfect neighbourhood set* (abbreviated to PN-set) if $\phi(X) = \bigcup_{x \in X} pn(x, X) \succ V$. Observe that each of these three types of sets may be characterized in terms of the partition $\mathcal{P}(X)$, since a set X is a packing, an irredundant set or a PN-set if and only if $C \cup Y = \emptyset$, $E \cap N(y) \neq \emptyset$ for each $y \in Y$ and $Z \cup E \cup F \succ V$ (respectively). In order to motivate the definition of the fourth principal property, we first state a condition given in [5] for an irredundant set to be maximal. We need one additional concept about private neighbourhoods. For $x \in X$ and $u \in V - X$, *u annihilates x* (or *x is annihilated by u*) if $\emptyset \neq pn(x, X) \subseteq N[u]$. Observe that if u annihilates x, then $pn(x, X \cup \{u\}) = \emptyset$, *i.e.* (informally) addition of u to X destroys (or annihilates) the private neighbourhood of x. Let

$$A_X = \{u \in V - X \mid u \text{ annihilates some } x \in X\}.$$

We write A for A_X, if the basic subset X is clear. For $U \subseteq V - X$ define X to be *U-annihilated* if $U \subseteq A$. We can now state a condition for an irredundant set X to be maximal in terms of the partition $\mathcal{P}(X)$.

Theorem 1 [5]. *The set X is maximal irredundant if and only if X is irredundant and $N[R]$-annihilated.*

We observe that the class of $N[R]$-annihilated sets (such sets have also been called *external redundant sets* ([5,6])) is contained in the larger class of R-annihilated sets (abbreviated to Ra-sets), which is the fourth class of sets of principal interest in this work. We will also consider sets which are both R-annihilated and irredundant (abbreviated to Rai-sets) which were first introduced by Favaron and Puech [11] (who called them *semi-maximal irredundant sets*). Notice that for each $z \in Z$ and $r \in R$, $z \in pn(z, X) - N[r]$ and so z is not annihilated by r. Thus any vertex of X which is annihilated by $r \in R$, is necessarily in Y. The main parameters considered are $\theta(G)$, $\theta_i(G)$, $ra(G)$, $rai(G)$, $ir(G)$ which are the smallest cardinalities of PN-sets, independent PN-sets, Ra-sets, Rai-sets and maximal irredundant sets respectively, and $\rho_L(G)$ ($\rho(G)$) which is the smallest (largest) cardinality of a maximal packing. Observe that a maximal packing is an independent PN–set. We will also mention $er(G)$, $\gamma(G)$, $\gamma_t(G)$, $\gamma_c(G)$, *i.e.* the smallest cardinalities of external redundant, dominating, total dominating and connected dominating sets respectively and $\gamma_2(G)$ which is the smallest cardinality of X such that each vertex of $V-X$ is within distance two of X. We abbreviate $\gamma(G)$ to γ etc. when the graph G involved is clear. Further, for example, a dominating set (maximal irredundant set) of minimum cardinality $\gamma(G)$ ($ir(G)$) will be called a *γ–set* (*an ir–set*). The following inequalities are immediately implied by the definitions, Theorem 1, the well-known inequality $ir \leq \gamma$, and the fact that for any two distinct vertices x and y of a packing X, the sets $N[x]$ and $N[y]$ are disjoint, while every dominating set must contain at least one vertex in $N[x]$ for each $x \in X$.

Proposition 2. *For any connected graph G,*

$$\gamma_2 \leq \left\{ \begin{array}{c} ra \leq \left\{ \begin{array}{c} rai \\ er \end{array} \right\} \leq ir \\ \theta \leq \theta_i \leq \rho_L \leq \rho \end{array} \right\} \leq \gamma \leq \gamma_t \leq \gamma_c,$$

where the last inequality holds only if $\Delta(G) < |V(G)| - 1$.

The motivation for this paper is threefold. Firstly, in [12] the authors conjectured that for any graph, $\theta \leq ir$. This conjecture was proved to be false by Favaron and Puech [11] who constucted counterexamples, the smallest of which has roughly two million vertices. However, the inequality has been established for several classes of graphs. For example, Cockayne *et al.* [9] showed that $\theta \leq ir$ for any tree. The second motivation for this work is the observation by Favaron and Puech [11] that the proof in [9] used only R-annihilation rather than $N[R]$-annihilation and independent PN–sets rather than PN–sets, and hence the same proof establishes the following stronger result.

Theorem 3 [9]. *For any tree, $\theta_i \leq rai$.*

Thirdly, the concept of irredundance has not yet been very well understood and remains difficult to work with. By studying the partition $\mathcal{P}(X)$ induced by an irredundant set X and the various ways in which $x \in X$ can be annihilated, we hope to gain a more thorough understanding of this fascinating concept.

Note that in many graphs $rai < ir$. For example, the graph consisting of two copies of C_4 joined by an edge has $rai = 2$ and $ir = 3$. Puech [15] has established the stronger inequality $\theta_i \leq rai$ for other classes of graphs. Let T be the tree consisting of disjoint copies of P_3 and $\overline{K}_3$ joined by a matching.

Theorem 4 [15]. *If G has no induced subgraph isomorphic to T, then $\theta = \theta_i \leq rai$.*

Theorem 5 [15]. *If G is chordal or if G contains at most one cycle of length different from* 3,4,7,8,9,13,14,19, *then $\theta_i \leq rai$.*

In Section 2 we indicate some other known results about ir which may be strengthened to results concerning R-annihilated sets. Section 3 gives various new degree conditions on G which will ensure $\theta_i \leq rai$ and also in two cases, conditions which imply the stronger inequality $\rho_L \leq rai$. The fact that the latter inequality holds for trees will be established in [3]. The final result relates ρ and ra for any graph G. Extremal graphs for the inequalities considered in this paper will be discussed in forthcoming work. References to further work on domination, irredundance and packing may be found in the comprehensive bibliography of the book by Haynes, Hedetniemi and Slater [13]. Perfect neighbourhood sets were introduced and studied in [12]. Other properties of Rai-sets were discussed by Puech [15]. In particular he showed that for any graph G,

$$0 < \frac{\theta_i}{rai} \leq \frac{3}{2}.$$

This result will be improved in Section 3.

2. Strengthening of Existing Results

In this section we use the fact that proofs of several existing results about ir only require the R-annihilation property and do not require $N[R]$-annihilation or irredundance. Hence the same arguments establish stronger results. Since these proofs are already in the literature, we omit them here.

Theorem 6 [1].

(i) *If the subgraph induced by an ir-set has k ($< ir$) isolated vertices, then $\gamma \leq 2ir - (k+1)$.*

(ii) *For any connected graph, $\gamma_t \leq 2ir$.*

The same methods establish:

Theorem 7.

(i) *If the subgraph induced by an ra-set has k isolated vertices, then $\gamma \leq 2ra - k$.*

(ii) *For any connected graph, $\gamma_t \leq 2ra$.*

The following relationship was proved by Favaron and Kratsch.

Theorem 8 [10]. *If G is connected, then $\gamma_c \leq 3ir - 2$.*

The same proof in fact shows

Theorem 9. *If G is connected, then $\gamma_c \leq 3ra - 2$.*

The rest of this section concerns lower bounds for parameters in terms of the number of vertices n and the maximum degree Δ. Bollobás and Cockayne established

Theorem 10 [2]. *For any graph G, $ir \geq \frac{n}{2\Delta - 1}$.*

Extremal graphs for this inequality were characterized by Laskar and Pfaff [14]. The proof of Theorem 10 also shows

Theorem 11. *For any graph G, $ra \geq \frac{n}{2\Delta - 1}$.*

In [8], Cockayne and Mynhardt improved the bound of Theorem 10.

Theorem 12 [8]. *For any graph G, $ir \geq \frac{2n}{3\Delta}$.*

The extremal graphs for this inequality were also characterized. The proof of Theorem 12 does not use irredundance, but does require $N[R] \cap (R \cup E)$-annihilation (observed in [5]). If $\eta = \eta(G)$ is the minimum cardinality of an $N[R] \cap (R \cup E)$–annihilated set of G, then (from the definition) $ra \leq \eta \leq er \leq ir$. The proof of Theorem 12 also establishes the stronger result:

Theorem 13. *For any graph G, $\eta \geq \frac{2n}{3\Delta}$.*

3. Degree Conditions, θ_i, ρ_L, ρ and rai

For the work of this section up to and including Theorem 22, X will denote an Rai–set (sometimes of minimum cardinality rai). We need extra notation. For the vertex u and vertex subset U of G, $d_U(u)$ denotes the number of edges of G from u to U. Define $B = E \cup F$ and for $x \in X$ let $B_x = N(x) \cap B$. By D_k, D_k^+, we denote the sets of vertices with degree equal to k and at least k, respectively. Further, define

$$S = \{y \in Y \mid d_C(y) \geq 1\},$$

$$T = \{y \in Y - S \mid d_S(y) \geq 2\}$$

$$\text{and} \quad \overline{T} = T \cap D_4^+.$$

It is not difficult to prove (see Cockayne, Favaron, Mynhardt and Puech [4]) that if D_3^+ is independent, then $\gamma(G) = i(G)$. We show here that if D_4^+ is independent, then $\theta_i(G) \leq rai(G)$. We begin with a lemma.

Lemma 14. *If D_4^+ is independent, then each component of $G[S \cup \overline{T}]$ is isomorphic to $K_{1,n}$ $(n \geq 0)$ with central vertex in D_4^+ or a copy of K_1 or K_2 in $S - D_4^+$.*

Proof: For any $s \in S$, $d_C(s) \geq 1$ (definition of S), $d_B(s) \geq 1$ (since $s \in S \subseteq Y$ has an epn) and $d_X(s) \geq 1$ (since $s \in Y$). If $\deg(s) = 3$, then it follows that $d_C(s) = d_B(s) = d_X(s) = 1$. Hence

$$\text{for each } s \in S, \ \deg(s) \geq 3 \text{ and if } \deg(s) = 3, \text{ then } d_X(s) = 1. \qquad (1)$$

Let Ω be a component of $G[S \cup \overline{T}]$. Firstly, suppose that Ω contains $y_\Omega \in D_4^+$. The independence of D_4^+ implies that if $w \in N(y_\Omega) \cap \Omega$, then $w \in S$ and has degree at most three. By (1), $\deg(w) = 3$ and y_Ω is the only neighbour of w in Ω. Thus Ω is isomorphic to $K_{1,n}$ ($n \geq 0$) centred at $y_\Omega \in D_4^+$. Otherwise Ω is contained in $S - D_4^+$ and by (1), Ω is isomorphic to K_1 or K_2. ▮

Theorem 15. *If D_4^+ is independent, then $\theta_i \leq rai$.*

Proof: Let X be an rai–set and recall Lemma 14. If the component Ω of $G[S \cup \overline{T}]$ is a star, let y_Ω be its centre. Otherwise Ω is K_1 with vertex y_Ω. The independent set

$$A = \left\{ y_\Omega \mid \Omega \text{ is a component of } G[S \cup \overline{T}] \right\},$$

is a packing of $G^* = G[V - C_A]$. Definitions show that $C_A \subseteq C \cup (Y - (S \cup \overline{T}))$. Embed A in a maximal packing $\overline{A}$ of G^*. Since $A \subseteq \phi(\overline{A})$ and $A \succ C_A$, $\overline{A}$ is an independent PN–set of G and $\overline{A} \cap C_A = \emptyset$. Also note that no vertex of C_A has two or more X–pns in $\overline{A}$. The definitions of A and S imply that each vertex of $C \cap N(Y)$ is at distance at most two from A and so is not in $\overline{A}$. Hence $C \cap \overline{A} \subseteq N(Z)$. Further, since X is an Rai-set, each $r \in R$ annihilates at least one $y \in Y$. It follows that we may define a function $f : \overline{A} \to X$ by:

$$f(\bar{a}) = \begin{cases} \text{an arbitrary vertex of } N(\bar{a}) \cap Z & \text{if} \quad \bar{a} \in C \\ \bar{a} & \text{if} \quad \bar{a} \in X \\ \text{the unique } x \in X \text{ such that } \bar{a} \in B_x & \text{if} \quad \bar{a} \in B \\ \text{an arbitrary vertex in } Y \text{ annihilated by } \bar{a} & \text{if} \quad \bar{a} \in R. \end{cases}$$

We now show that f is injective. Suppose to the contrary that there exist $a, b \in \overline{A}$ with

$$y = f(a) = f(b). \tag{2}$$

Since $\overline{A}$ is a packing of G^*, $y \in C_A$. Since $C_A \subseteq C \cup \left(Y - (S \cup \overline{T})\right)$, the definition of f implies that $y \in C_A \cap Y$. This fact and $\overline{A} \cap C_A = \emptyset$, together with the packing property of $\overline{A}$ and the definition of f, imply that neither a nor b is in $C \cup X$ and hence $\{a, b\} \subseteq R \cup B$. If both a and b are in R, (2) implies that each annihilates y. If $a \in R$ and $b \in B$, (2) implies that ab is an edge of G^*. In both cases the packing property of $\overline{A}$ is contradicted. It remains to show that $\{a, b\} \subseteq B$ is impossible. In this case for y defined by (2), $|B_y| \geq 2$. Since $y \in C_A$, $d_X(y) \geq 2$ and so $y \in D_4^+$. Recall that $A \subseteq S \cup \overline{T}$ where $\overline{T} \subseteq D_4^+$. The independence of D_4^+ implies that $y \notin N(A \cap \overline{T})$ and therefore y is adjacent to at least two vertices of $A \cap S$. It follows that $y \in T \cap D_4^+ = \overline{T}$. However, $C_A \cap \overline{T} = \emptyset$, a contradiction. We have proved that f is injective and so $|\overline{A}| \leq |X|$. Since $\overline{A}$ is an independent PN-set of G and X is a rai-set, $\theta_i \leq rai$ as asserted. ▮

Corollary 16. *If D_4^+ is independent, then $\theta \leq \theta_i \leq rai \leq ir$.*

Two more preliminary results are now necessary.

Lemma 17. *If D_4^+ is a packing and D_3 is independent, then $T = \overline{T}$ and each component Ω of $G[S \cup T]$ is a copy of $K_{1,n}$ $(n \geq 0)$ centred in D_4^+ or a K_1 $(= K_{1,0})$ in $S - D_4^+$.*

Proof: Suppose to the contrary that there exists $t \in T - \overline{T}$. By definition of T, t has neighbours s_1 and s_2 in S. Now $\deg(t) \geq d_B(t) + d_S(t) \geq 3$. But $t \notin D_4^+$ and so $\deg(t) = 3$. For $i = 1, 2$, $\deg(s_i) \geq d_C(s_i) + d_T(s_i) + d_B(s_i) \geq 3$. However, D_3 is independent and so $s_i \in D_4^+$. But s_1, s_2 have the common neighbour t which implies that D_4^+ is not a packing, contrary to hypothesis. Since D_4^+ is independent, the conclusion of Lemma 14 is true. Suppose that Ω is a copy of K_2 in $S - D_4^+$, with vertices s_1, s_2. Now for each $i = 1, 2$, $\deg(s_i) \geq 3$. But $s_i \notin D_4^+$ and so $\deg(s_i) = 3$, contrary to the independence of D_3. The result now follows from Lemma 14. ∎

Lemma 18. *Let $\overline{C} = \{c \in C \mid d_Z(c) = 0\}$ and P be a packing of G such that $P \cap \overline{C} = \emptyset$. Then $|P| \leq |X|$.*

Proof: We define the relation $f : P \to X$ by

$$f(p) = \begin{cases} \text{any vertex of } N(p) \cap Z & \text{if} \quad p \in C - \overline{C} \\ \text{the unique } x \in X \text{ such that } p \in B_x & \text{if} \quad p \in B \\ p & \text{if} \quad p \in X \\ \text{any } x \in Y \text{ such that } p \text{ annihilates } x & \text{if} \quad p \in R. \end{cases}$$

The hypothesis implies that f is a well-defined injective function so that $|P| \leq |X|$. ∎

Theorem 19. *If D_4^+ is a packing and D_3 is independent, then $\rho_L \leq rai$.*

Proof: Let X be an rai-set and recall Lemma 17. For the component Ω of $G[S \cup T]$ $(= G[S \cup \overline{T}])$ let y_Ω be the central vertex of Ω and $A = \{y_\Omega \mid \Omega$ is a component of $G[S \cup T]\}$. Since D_4^+ is a packing, $A \cap D_4^+$ is also a packing. Choose Q, a maximal subset of A such that Q is a packing of G containing $A \cap D_4^+$ and extend Q to a maximal packing P of G. We show that $P \cap \overline{C} = \emptyset$. Suppose to the contrary that there exists $c \in P \cap \overline{C}$. There are two cases to consider which depend on Lemma 17.

Case 1. Suppose there exists $s \in N(c) \cap S$ such that the component Ω of $G[S \cup T]$ containing s, is a $K_{1,n}$ $(n \geq 0)$ centred at $y_\Omega \in D_4^+$. In this case $y_\Omega \in A \cap D_4^+ \subseteq P$. However, $s \in N[c] \cap N[y_\Omega]$ which contradicts the packing property.

Case 2. Each $s \in N(c) \cap S$ is an isolated vertex of $G[S \cup T]$ and has degree three. Note that each such $s \in A$ satisfies $d_B(s) = d_C(s) = 1$ and s is adjacent to precisely one vertex y of $Y - (S \cup T)$. Since $s \in D_3$, the independence of D_3 and the fact that $pn(y, X) \neq \emptyset$ imply that $y \in D_2 \cup D_4^+$. We claim that at least one of the vertices s of $N(c) \cap S$ is in the packing Q. For otherwise select any such s and let q be an arbitrary element of Q $(\subseteq A)$. Since $q \notin N(c)$, $N(q) \cap N(s) \subseteq \{y\}$. If $q \in D_4^+$ and y is adjacent to q, then $y \in D_4^+$, contradicting the independence of D_4^+. Hence y is not adjacent to q and so $N[s] \cap N[q] = \emptyset$. If $q \in A - D_4^+$, then by Lemma

17, $q \in S$, thus y is not adjacent to q (otherwise $y \in T$). Since $q \notin N(c)$ it again follows that $N[s] \cap N[q] = \emptyset$ and so $Q \cup \{s\}$ is a packing of G, which contradicts the maximality of Q. Thus some $s \in N(c) \cap S$ satisfies $s \in Q \subseteq P$, a contradiction with $c \in P$. Cases 1 and 2 assert that $P \cap \overline{C} = \emptyset$ and the theorem now follows from Lemma 18. ▮

Corollary 20. *If D_4^+ is a packing and D_3 is independent, then $\theta \leq \theta_i \leq \rho_L \leq rai \leq ir$.*

Lemma 21. *If $D_4^+ = \emptyset$ (i.e. $\Delta \leq 3$), then any component Ω of $G[S \cup T]$ is a copy of:*

P_3 with vertex sequence $s_1 t_\Omega s_2$ where $t_\Omega \in T$ and $\{s_1, s_2\} \subseteq S$,

P_2 with vertices s_1 and s_2 both in S

or P_1 with vertex $s \in S$.

Proof: The definitions of S and T imply that each $s \in S$ and $t \in T$ have degree three and

$$\left.\begin{array}{l} d_X(s) = d_C(s) = d_B(s) = 1 \\ d_{X-S}(t) = 0, \quad d_S(t) = 2, \quad d_B(t) = 1. \end{array}\right\} \tag{3}$$

If Ω is a single vertex s, then (3) implies that $s \in S$. If Ω contains $t_\Omega \in T$, then (3) asserts that Ω is a P_3 with vertex sequence $s_1 t_\Omega s_2$ with s_1 and s_2 both in S. Otherwise Ω is contained in S and (by (3)) is a P_2. ▮

Theorem 22. *If $D_4^+ = \emptyset$, then $\rho_L \leq rai$.*

Proof: Suppose that X is an *rai*-set. Recall Lemma 21, observe that $T = \{t_\Omega \mid \Omega \text{ is isomorphic to } P_3\}$ and let K be the set of all vertices of P_1 or P_2 components of $G[S \cup T]$. Observe that since $d_{X-S}(t) = 0$, T is a packing of G. Let Q be a maximal subset of $T \cup K$ such that Q is a packing of G which contains T and extend Q to a maximal packing P of G. We show that $P \cap \overline{C} = \emptyset$. Suppose to the contrary that there exists $c \in P \cap \overline{C}$. There are two situations which depend on Lemma 21. If there exists $s \in N(c) \cap S$ such that s is in Ω, a P_3 component of $G[S \cup T]$, then $t_\Omega \in T \subseteq P$. However, $s \in N[c] \cap N[t_\Omega]$ which contradicts the packing property. Otherwise $N(c) \cap S \subseteq K$. Choose any $s \in N(c) \cap S$ and let $N(s) \cap Y = \{y\}$. Since $c \in P$,

$$c \notin N[Q], \quad s \notin N[Q] \quad \text{and} \quad y \notin Q. \tag{4}$$

But by (3), $N(y) \cap (S \cup T) = \{s\}$ and so $y \notin N(Q)$. This, together with (4), implies that $N[s] \cap N[Q] = \emptyset$. Hence $Q \cup \{s\}$ is a packing of G contrary to maximality. Thus $P \cap \overline{C} = \emptyset$ and the theorem follows from Lemma 18. ▮

Corollary 23. *If $\Delta(G) \leq 3$, then $\theta \leq \theta_i \leq \rho_L \leq rai \leq ir$.*

Our last result improves a result of Puech [15]. Let k be the largest number of isolated vertices in the induced subgraph of any *ra*-set of G.

Theorem 24. *For any graph G, $\rho \leq \left\lfloor \frac{3ra-k}{2} \right\rfloor$.*

Proof: Let X be an ra-set of G such that $G[X]$ has k isolates (*i.e.* $|Z| = k$) and P be a maximum packing (of cardinality ρ) of G. The R-annihilation property implies that we may partition $X \cup B \cup R$ into

$$\bigcup_{x \in X} \{x\} \cup B_x \cup R_x \quad \text{(disjoint union)}$$

where for each $x \in X$, R_x is a subset of vertices of R which annihilate x. Note that $R_z = \emptyset$ for $z \in Z$. Since $G[\{x\} \cup B_x \cup R_x]$ has diameter at most two, each $\{x\} \cup B_x \cup R_x$ contains at most one element of P. Suppose that q $(\leq k)$ is the number of vertices z of Z such that $\{z\} \cup B_z \cup R_z$ (in fact $R_z = \emptyset$) contains exactly one vertex of P. For each such z, by the packing property, z is not adjacent to $C \cap P$. Further, the vertices of $C \cap P$ are adjacent to disjoint subsets of X of size at least two. Hence

$$|P \cap C| \leq \frac{|X| - q}{2}. \tag{5}$$

But there are at least $k - q$ vertices of Z for which $P \cap (\{x\} \cup B_x \cup R_x) = \emptyset$ and so

$$|P \cap (X \cup B \cup R)| \leq |X| - (k - q). \tag{6}$$

From (5), (6) and $q \leq k$ we deduce that $|P| \leq \frac{3|X| - k}{2}$ and the result follows. ∎

Corollary 25 [14]. *For any graph G, $\theta_i \leq \frac{3rai}{2}$.*

Proof: Immediate deduction from Theorem 24 and Proposition 2. ∎

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