Minimal bases for the laws of certain cycle systems

Sarah Zahrai*

Centre for Discrete Mathematics and Computing
The University of Queensland, Queensland 4072, Australia

Abstract

In this paper we show that the varieties of groupoids arising from 2-perfect 5 and 7-cycle systems and also those arising from strongly 2-perfect 5 and 7-cycle systems can be defined by two laws.

1 Introduction

The search for single axioms for varieties of groupoids has long interested mathematicians and there are many such axiomatizations, such as for groups [6], lattices [10] and sloops and squags [4]. One of the interesting open cases is that of groupoids related to cycle systems. In a paper by D. Bryant and C. Lindner [2], it is shown that the classes of groupoids arising from 2-perfect m-cycle systems can be equationally defined as quasigroups for $m = 3, 5$, and 7 only and bases for each of these, consisting of three laws, are given. In [3] D. Bryant and S. Oates-Williams give laws for groupoids arising from strongly 2-perfect cycle systems, under some special conditions on m. In this paper we show that the varieties of groupoids arising from 2-perfect 5 and 7-cycle systems, and also those arising from strongly 2-perfect 5 and 7-cycle systems, can be equationally defined as quasigroups by two laws. Moreover, we show that the two laws in the strong cases force the groupoids in question to be quasigroups.

2 Definitions and Preliminary results

First we need the concept of a cycle system—a generalisation of Steiner triple systems.

Definition 2.1 Let K_n be the complete graph on n vertices, then an *m-cycle system of order n* is a decomposition of K_n into edge disjoint cycles of length m.

*This work was done while the author held a scholarship funded by the ARC.

We shall abbreviate m-cycle system to m-CS, and, if S is the set of vertices of K_n and C the set of cycles in the m-CS, we shall denote the system by (S, C).

We also need to define a binary operation on the cycle systems to get a groupoid.

Definition 2.2 *(Standard Construction)* Let (S, C) be an m-cycle system and define a binary operation $\circ$ on S by:

(1) $x \circ x = x$, for all $x \in S$;

(2) if $x \neq y, x \circ y = z$ and $y \circ x = w$ if and only if $(\cdots w, x, y, z, \cdots) \in C$.

From now on we use the binary operation obtained from the standard construction in our laws, although for clarity of exposition we omit "$\circ$" throughout the paper.

In general, the standard construction does not yield a quasigroup; a stronger condition is required.

Definition 2.3 [1] Let c be a cycle of length m and $c(i)$ denote the graph formed from c by joining all vertices at distance i in c. If (S, C) is an m-cycle system of K_n such that $(S, \{c(i) \mid c \in C\})$ is also a cycle system of K_n, then we call (S, C) an *i-perfect* m-cycle system.

(Note that this means that in a i-perfect m-cycle system, (S, C), every pair of vertices is joined by a path of length i in precisely one cycle of C.)

In [2] Bryant and Lindner give collections of identities which define quasigroups arising from m-CSs under the standard construction. In general, the class of 2-perfect m-CSs is said to be *equationally defined* provided there exists a collection of identities I such that a finite quasigroup belongs to the variety $V(I)$ if and only if it can be constructed from a 2-perfect m-CS using the Standard Construction. The following theorems show that 2-perfect 3-cycle systems (Steiner Triple Systems), 2-perfect 5-cycle systems and 2-perfect 7-cycle systems are equationally defined by three identities. [Note that the quasigroups defined by Steiner Triple Systems are called *squags*.]

Theorem 2.4 [5] *A basis for the variety of all squags is given by:*

(1) $xx = x$,

(2) $xy = yx$, *and*

(3) $x(xy) = y$. □

Theorem 2.5 [7] *2-perfect 5-cycle systems are equationally defined by the identities:*

(1) $x^2 = x$,

(2) $(yx)x = y$, *and*

(3) $x(yx) = y(xy)$. □

Theorem 2.6 [7] *2-perfect 7-cycle systems are equationally defined by the identities:*

(1) $x^2 = x$,

(2) $(yx)x = y$, *and*

(3) $(xy)(y(xy)) = (yx)(x(yx))$. □

It is known that $m = 3, 5$ and 7 are the only values of m for which 2-perfect m-cycle systems can be equationally defined (see [2]), but in [3] by placing additional restrictions on the 2-perfect systems, more equationally defined systems are obtained. This new condition is defined as follows:

Definition 2.7 A 2-perfect m-cycle system is said to be *strongly 2-perfect* if the distance two cycles also belong to the system.

Now this extra condition leads us to the following theorem.

Theorem 2.8 [3]. *Let m be*
(a) an odd prime less than 127; or
(b) a prime greater than 127 for which either
(i) 2 has order $m-1 \pmod{m}$; or
(ii) 2 has order $\frac{m-1}{2}$ and $m \equiv 3 \pmod{4}$.
Then the quasigroups corresponding to strongly 2-perfect m-cycle systems form a variety defined by the laws:
$W_2(x, x) = x$,
$W_2(W_2(x, y), y) = x$,
$W_m(x, y) = x$,
$W_{m+1}(x, y) = y$, *and*
$W_2(W_0(x, y), W_2(x, y)) = W_4(x, y)$,
where the sequence of words $W_i(x, y)$ is defined inductively by:
$W_0(x, y) = x$,
$W_1(x, y) = y$, *and*
$W_i(x, y) = W_{i-2}(x, y) \circ W_{i-1}(x, y)$. □

3 Main results

In [4] it is shown that the variety of all Steiner loops (sloops) and the variety of all Steiner quasigroups (squags) can be defined by single laws. In the following we show that the variety of 2-perfect 5-cycle systems and also the variety of 2-perfect 7-cycle systems can be defined by just two identities. To prove the following theorem and also the other results in this section we will use a series of substitutions and simplifications. The automated theorem-proving program Otter (see [9]) was used in a substantial way to obtain the results of this paper.

Theorem 3.1 *The variety of 2-perfect 5-cycle systems can be equationally defined by the following two laws:*
(i) $xx = x$,
(ii) $((xz)z)(yx) = ((yu)u)(xy)$.

Proof. We need to show that (i) and (ii) together are equivalent to the three laws in Theorem 2.5. It is easy to see that (1), (2) and (3) imply (i) and (ii). So we just need to show that from (i) and (ii) we can also get the other three laws.

Let $x = y = z$ in (ii), then using (i) we have

$$x = ((xu)u)x. \tag{4}$$

Putting $x = (xz)z$ in (4) we get

$$(xz)z = ((((xz)z)u)u)((xz)z).$$

Now let $x = u$, then

$$(xz)z = ((((xz)z)x)x)((xz)z)$$

and (4) and (i) imply

$$(xz)z = x((xz)z). \tag{5}$$

Now let $y = (xz)z$ in (ii):

$$((xz)z)(((xz)z)x) = ((((xz)z)u)u)(x((xz)z)).$$

Then putting $x = u$ and applying (4) and (i) we get

$$x = x(x((xz)z))$$

and (5) implies

$$x = (xz)z$$

and (2) holds.
To get (3) we just need to apply (2) to (ii), then

$$x(yx) = y(xy)$$

as required. □

In the following we state our other results. The proofs of Theorem 3.2, 3.3(a) and 3.4(a) are obtained in the same way as that of Theorem 3.1. As these proofs are very lengthy, they have been omitted. The reader may verify the statements using Otter, or the author can supply proofs on request. The proofs for 3.3(b) and and 3.4(b) are given, since Otter is not applicable here.

Theorem 3.2 *The variety of 2-perfect 7-cycle systems can be defined by the following two laws:*
(i) $xx = x$,
(ii) $(yx)(((xz)z)(((yv)v)x)) = (((xw)w)y)(((yu)u)(xy))$. □

Theorem 3.3 *(a) The variety of strongly 2-perfect 5-cycle systems can be defined by the following two laws:*
(i) $xx = x$,
(ii) $((xz)z)(yx) = (u(u(yu)))(xy)$.
(b) Moreover, groupoids defined by these two laws are quasigroups.

Proof. We need to show that (i) and (ii) are equivalent to the following identities
(1) $xx = x$,
(2) $x(yx) = y(xy)$,
(3) $(yx)x = y$, and
(4) $y(y(xy)) = x$ (Strong condition),

but as we mentioned before, we will just prove the second part of this theorem, that is, the quasigroup conditions. To do this we define the following multiplications:
$xL(a) = ax$ (left multiplication)
$xR(a) = xa$ (right multiplication).
Now, by rewriting (3) we will have

$$R(x)R(x) = I,$$

so $R(x)$ is one to one and also onto, and hence it is a permutation.
Using these operations for (4) we get

$$R(y)L(y)L(y) = I$$

and $L(y)$ is also a permutation.
As left and right multiplications are permutations, the groupoid is a quasigroup. □

Theorem 3.4 *(a) The variety of strongly 2-perfect 7-cycle systems can be defined by the following two laws:*
(i) $xx = x$,
(ii) $(yx)(((xz)z)((v((vy)(y(vy))))x)) = (((xw)w)y)((u((uy)(y(uy))))(xy))$.
(b) Moreover, these two laws make the groupoids related to 7-cycle systems into quasigroups.

Proof. Again we can show that (i) and (ii) are equivalent to the following equations:
(1) $xx = x$,
(2) $(xy)y = x$,
(3) $(xy)(y(xy)) = (yx)(x(yx))$, and
(4) $y((yx)(x(yx))) = x$.

From these laws we can also get the following extra law which helps us to prove the quasigroup conditions

(5) $x = y(y(yx))$.

Here again we use left and right multiplications, but this time on (2) and (5).
Equation (2) becomes

$$R(y)R(y) = I,$$

so $R(y)$ is one to one and onto and hence is a permutation.
Equation (5) can also be written as

$$L(y)L(y)L(y) = I,$$

so $L(y)$ is a permutation.
Hence the groupoids arising from 2-perfect 7-cycle systems are quasigroups. □

4 Conclusion

Unfortunately, we have not been able to emulate the results of [4], [6] and [10] by producing a single law for 5-cycle and 7-cycle systems, neither in the ordinary nor the strong case. However, in a later paper, we shall give single laws for a new type of cycle system recently introduced by C. Lindner and C. Rodger [8].

References

[1] P. Adams and E.J. Billington, *Completing some spectra for 2-perfect cycle systems*, Australasian Journal of Combinatorics, **7** (1993), 175–187.

[2] D.E. Bryant and C.C. Lindner, *2-perfect m-cycle systems can be equationally defined for $m = 3$, 5 and 7 only*, Algebra Universalis, **35** (1996), 1–7.

[3] D.E. Bryant and S. Oates-Williams, *Strongly 2-perfect cycle systems and their quasigroups*, 15th British Combinatorial Conference (Stirling, 1995). Discrete Math, **167/168** (1997), 167–174.

[4] D. M. Donovan and S. Oates-Williams, *Single laws for sloops and squags*, Discrete Math, **92** (1991), 79–83.

[5] B.Ganter and H. Werner, *Co-ordinatizing Steiner systems*, Discrete Math, **7** (North-Holland, Amsterdam, 1980), 3–24.

[6] G. Higman and B. H. Neumann, *Groups as groupoids with one law*, Publicationes Mathematicae Debrecen, **2** (1952), 215–227.

[7] C. C. Lindner, *2-perfect m-cycle systems and quasigroup varieties : A Survey*, 24th annual Iranian mathematics conference, Tehran, 1993.

[8] C. C. Lindner and C. A. Rodger, *On equationally defining extended cycle systems*, Discrete Math, **173** (1997), 1–14.

[9] W. McCune, Otter 3.0 Reference Manual and Guide. Tech. Report ANL-94/6, Argonne National Laboratory, Argonne, IL, (1994).
See also `http://www.mcs.anl.gov/home/mccune/ar/otter`.

[10] W. McCune and R. Padmanabhan, *Single identities for lattice theory and for weakly associative lattices*, Algebra Universalis, **36** (1996), 436–449.

(Received 29/7/97; revised 13/1/98)