Minus k-subdomination in graphs III

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Abstract

Let $G = (V, E)$ be a graph. For any real valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The weight of f is defined as $f(V)$. We will also denote $f(N[v])$ by $f[v]$, where $v \in V$. A minus k-subdominating function (kSF) for G is defined in [1] as a function $f : V \to \{-1, 0, 1\}$ such that $f[v] \geq 1$ for at least k vertices of G. The minus k-subdomination number of a graph G, denoted by $\gamma_{ks}^{-101}(G)$, is equal to $\min\{f(V) \mid f \text{ is a minus } kSF \text{ of } G\}$. Hattingh and Ungerer show in [5] that if T is a tree of order $n \geq 2$ and k is an integer such that $1 \leq k \leq n-1$, then $\gamma_{ks}^{-101}(T) \geq k - n + 2$. In this paper, we characterise trees which achieve the lower bound, and show that the decision problem corresponding to the computation of this parameter is NP-complete.

1 Introduction

Let $G = (V, E)$ be a graph and let v be a vertex in V. The *open neighbourhood* of v is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u | uv \in E\}$. The *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$.

For any real valued function $f : V \to \mathbf{R}$ and $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The *weight* of f is defined as $f(V)$. We will also denote $f(N[v])$ by $f[v]$, where $v \in V$.

A *minus dominating function* is defined in [3] as a function $f : V \to \{-1, 0, 1\}$ such that $f[v] \geq 1$ for every $v \in V$. The *minus domination number* of a graph G is $\gamma^-(G) = \min\{f(V) \mid f$ is a minus dominating function on $G\}$. A *minus k-subdomination function* (kSF) for G is defined in [1] as a function $f : V \to \{-1, 0, 1\}$ such that $f[v] \geq 1$ for at least k vertices of G. The *minus k-subdomination number* of a graph G, denoted by $\gamma_{ks}^{-101}(G)$, is equal to $\min\{f(V) \mid f$ is a minus kSF of $G\}$. Let f be a kSF for the graph G. The set of vertices *covered by* f is defined as $C_f = \{v \in V | f[v] \geq 1\}$, while the set P_f is defined as $\{v \in V | f(v) = 1\}$.

The motivation for studying these variations of the domination number is rich and varied from a modelling perspective. By assigning the values -1, 0 or $+1$ to the vertices of a graph, we can model negative or neutral responses of preferences in such things as political voting or social behaviours. By examining these parameters, we study situations in which, in spite of the presence of negative vertices, the closed neighbourhoods of at least k of the vertices are required to maintain a positive sum, i.e. at least k groups of voters vote positively.

A *remote vertex* v of a graph G is a vertex which is adjacent to an endvertex of G. Hattingh and Ungerer show in [5] that if T is a tree of order $n \geq 2$ and k is an integer such that $1 \leq k \leq n-1$, then $\gamma_{ks}^{-101}(T) \geq k-n+2$. In Section 1 of this paper, we characterise those trees which achieve this lower bound. Then, in Section 2, we show that the decision problem corresponding to the computation of this parameter is NP-complete.

2 The characterisation

Hattingh and Ungerer ([5]) established the following result.

Theorem 1 *[5] If T is a tree of order $n \geq 2$ and k is an integer such that $1 \leq k \leq n-1$, then*

$$\gamma_{ks}^{-101}(T) \geq k - n + 2.$$

Moreover, this bound is best possible.

However, trees which achieve the lower bound were not characterised in [5]. The following result provides a solution to this problem.

Theorem 2 *Let $n \geq 2$ and let $1 \leq k \leq n-1$ be an integer. Then, for a tree T of order n, $\gamma_{ks}^{-101}(T) = k - n + 2$ if and only if*

(a) T has a vertex v adjacent to at least k endvertices, or

(b) T has a vertex v with $\deg(v) = k$ and at least $k-1$ neighbours of v are endvertices, or

(c) T has two adjacent vertices u and v with $\deg(u) = 2$ and $\deg(v) = k-1$ where all the other neighbours of v are endvertices, or

(d) T has two adjacent vertices u and v with $\deg(u) + \deg(v) = k+1$ or $k+2$ such that u and v together are adjacent to at least $k-2$ endvertices, or

(e) T has a vertex w of degree three and two of the neighbours of w together are adjacent to exactly $k-3$ other vertices, all of which are endvertices.

Proof. Let T be a tree of order n such that $\gamma_{ks}^{-101}(T) = k - n + 2$ and let f be a kSF of T such that $f(V(T)) = \gamma_{ks}^{-101}(T)$. Let $M = \{v \in V(T) | f(v) = -1\}$, $Z = \{v \in V(T) | f(v) = 0\}$ and $P = \{v \in V(T) | f(v) = 1\}$. Note that, since $k \geq 1$, $P \neq \emptyset$. Before proceeding further, we prove that $|M| \geq n-k-1$. For suppose to the contrary that $|M| \leq n-k-2$. Then, using the fact that $|P| = f(V(T)) + |M|$, it follows that $|P| \leq (k-n+2)+(n-k-2) = 0$, which is a contradiction. Hence, $|M| = n-k+s$ where $s \geq -1$ is an integer. Furthermore, $|P| = (k-n+2)+(n-k+s) = s+2$. Also, since $|Z| = n - (|M|+|P|)$, we have $|Z| = n-(n-k+2s+2) = k-2s-2$.

We now show that $s \leq 0$. Let $M_c = C_f \cap M$, $Z_c = C_f \cap Z$ and $P_c = C_f \cap P$. Suppose $|M_c| = s+t$ where t is an integer and let $H = \langle M_c \cup P\rangle$. Then H is a forest, since T is a tree. Say H has ℓ components. Then $q(H) = p(H) - \ell = (s+t)+(s+2) - \ell = 2s+t+2-\ell \leq 2s+t+1$. Since $M_c \subseteq C_f$, each vertex of M_c must be adjacent to at least two vertices of P. Hence, $2s+2t = 2(s+t) = 2|M_c| \leq q(H) \leq 2s+t+1$, which implies that $t \leq 1$. Furthermore, since $k \leq |C_f| = |M_c|+|Z_c|+|P_c| = s+t+|Z_c|+|P_c|$, we have $k-s-t \leq |Z_c|+|P_c| \leq (k-2s-2)+(s+2) = k-s$, whence $t \geq 0$. Let $|P_c| = r$.

We first consider the case when $t = 0$. Then $|Z_c|+|P_c| \geq k-s$, so that $k-s-r \leq |Z_c| \leq k-2s-2$. Hence, $s+2 \leq r = |P_c| \leq |P| \leq s+2$, so that $|P_c| = s+2$, which implies that $P_c = P$. If $q(\langle P\rangle) = 0$, then there are no edges joining vertices in M_c to vertices in P; hence $|M_c| = s = 0$. If $q(\langle P\rangle) = 1$, say u and v are adjacent, then there can be one edge from M_c to u and one to v. But T is a tree, thus no vertex in M_c is adjacent to both u and v and thus $|M_c| = s = 0$, i.e. $M_c = \emptyset$.

We now consider the case when $t = 1$. Then $|Z_c|+|P_c| \geq k-s-1$, so that $k-s-r-1 \leq |Z_c| \leq k-2s-2$. Hence, $|P_c| = r \geq s+1$. If $q(\langle P\rangle) \geq 1$, then $q(H) \geq 1+2(s+1) = 2s+3$, contradicting the fact that $q(H) \leq 2s+2$. Hence, $\langle P\rangle \cong \overline{K}_{s+2}$. In this case each vertex in M_c must be adjacent to two vertices in P which gives $2s+4$ edges, a contradiction. Therefore, $M_c = \emptyset$ and $s = -1$.

Case 1. $s = -1$.

In this case $|M| = n-k-1$, $|Z| = k$ and $|P| = 1$. Let $P = \{v\}$. Then $M_c = \emptyset$ implies that $|Z_c| = k$ or $|Z_c| = k-1$ and $|P_c| = 1$. If $|Z_c| = k$, then $Z_c = Z$ and every vertex of Z is therefore adjacent to v and only v. Thus, case (a) occurs. If $|Z_c| = k-1$, and $P_c = P$, then each vertex of Z_c is adjacent to v, Z_c is an independent set and the vertex in $Z - Z_c$ is adjacent to exactly one vertex in $Z_c \cup \{v\}$. Furthermore, since $v \in C_f$, v is not adjacent to any of the vertices in M. Hence, each vertex in M is either adjacent to vertices in M or adjacent to the vertex in $Z - Z_c$. If the vertex in $Z - Z_c$ is adjacent to v, case (b) occurs. If the vertex $Z - Z_c$ is adjacent to exactly one vertex in Z_c, case (c) occurs.

Case 2 $s = 0$.

In this case $|M| = n-k$, $|Z| = k-2$, $|P| = 2$ and we must have $k \geq 2$. Since $M_c = \emptyset$, we have $Z = Z_c$ and $P = P_c$. Let $P = \{u, v\}$.

Case 2.1 $\langle P\rangle \cong K_2$.

Then u is adjacent to at most one vertex of M and the same is true for v. Since $Z = Z_c$, each vertex of Z must be adjacent to either u or v (but not both) and to no other vertex of T. Thus, case (d) occurs.

Case 2.2 $\langle P \rangle \cong \overline{K}_2$.

Since $\langle P \rangle \cong \overline{K}_2$, u and v are not adjacent to any of the vertices of M. Note that $M \neq \emptyset$, since $n - k \geq 1$. It follows that some vertex of M must be adjacent to some vertex of Z, say w. Since $w \in C_f$, w must be adjacent to both u and v. Note also that $|N[w] \cap M| = 1$. Since $Z_c = Z$, each vertex of $Z - \{w\}$ must be adjacent to u or v (but not both) and to no other vertex of T. Thus, case (e) occurs.

Conversely, the previous proof suggests in each case a kSF f of T such that $f(V(T)) = k - n + 2$. Also, Theorem 1 shows that $\gamma_{ks}^{-101}(T) \geq k - n + 2$. Our result follows. ■

This result supplements the following result of Dunbar, Hedetniemi, Henning and McRae (see [3]).

Theorem 3 *[3] If T is a tree, then $\gamma_{ns}^{-101}(T) \geq 1$. Furthermore, equality holds if and only if T is a star.*

3 Complexity results

Let $r \leq 1$ be a fixed positive rational number (in lowest terms). Consider the decision problem

PARTIAL MINUS DOMINATING FUNCTION (PMDF)

INSTANCE: A graph G and an integer ℓ.

QUESTION: Is there a function $f : V(G) \to \{-1, 0, 1\}$ of weight ℓ or less for G such that $|C_f| \geq r|V(G)|$?

In this section we show that **PMDF** is NP-complete by describing a polynomial transformation from the following NP-complete problem (see [4]):

EXACT COVER BY 3-SETS (X3C)

INSTANCE: A set $X = \{x_1, \ldots, x_{3q}\}$ and a set $\mathcal{C} = \{C_1, \ldots, C_m\}$ where $C_j \subseteq X$ and $|C_j| = 3$ for $j = 1, \ldots, m$.

QUESTION: Is there a subcollection $\mathcal{C}'$ of $\mathcal{C}$ such that each element of X occurs in exactly one member of $\mathcal{C}'$? ($\mathcal{C}'$ is an exact cover of X.)

If $r = 1$, then **PMDF** is the NP-complete problem **MINUS DOMINATING FUNCTION** (see [2]). Hence, we also assume that $r < 1$. For two real numbers a and b, we say that a *divides* b if there is an integer k such that $b = ka$.

Theorem 4 **PMDF** *is NP-complete, even for bipartite graphs.*

Proof. It is obvious that **PMDF** is in NP. To show that **PMDF** is an NP-complete problem, we will establish a polynomial transformation from the NP-complete problem **X3C**. Let $X = \{x_1, \ldots, x_{3q}\}$ and $\mathcal{C} = \{C_1, \ldots, C_m\}$ be an arbitrary instance of **X3C** where $C_j \subseteq X$ and $|C_j| = 3$ for $1 \leq j \leq m$. We will construct a bipartite graph G and an integer ℓ such that this instance of **X3C** will have an exact cover if and only if there is a function $f : V(G) \to \{-1, 0, 1\}$ of weight at most ℓ such that $|C_f| \geq r|V(G)|$.

The (bipartite) graph G is constructed as follows. Corresponding to each $x_i \in X$ associate the path x_i, w_i, v_i, u_i. Corresponding to each set C_j associate the path

c_j, d_j, e_j. The construction of G is completed by adding a set, denoted by U, of $\lceil \frac{3m+12q}{r} \rceil - (3m+12q)$ isolated vertices and the edges $\{x_ic_j | x_i \in C_j\}$. Lastly, set $\ell = 4m + 16q - 2\lfloor \frac{3m+12q}{r} \rfloor + \lceil \frac{3m+12q}{r} \rceil$. It is easy to see that the construction of G can be accomplished in polynomial time.

Before proceeding further, we prove

Claim 1 *If a and r are positive real numbers such that r does not divide a, then $0 < r - (a - r\lfloor \frac{a}{r} \rfloor) < r$, while $a + x \geq r\lceil \frac{a}{r} \rceil$ if and only if $x \geq r - (a - r\lfloor \frac{a}{r} \rfloor)$.*

Proof. Note that $a = r\lfloor \frac{a}{r} \rfloor + (a - r\lfloor \frac{a}{r} \rfloor)$. Since $a - r\lfloor \frac{a}{r} \rfloor$ is the remainder of a after division by r and a is not divisable by r, it follows that $0 < a - r\lfloor \frac{a}{r} \rfloor < r$, so that $0 < r - (a - r\lfloor \frac{a}{r} \rfloor) < r$.

Then $a + x \geq r\lceil \frac{a}{r} \rceil$ if and only if $x \geq r\lceil \frac{a}{r} \rceil - a$ if and only if $x \geq r\lceil \frac{a}{r} \rceil - r\lfloor \frac{a}{r} \rfloor - (a - r\lfloor \frac{a}{r} \rfloor)$ if and only if $x \geq r(\lceil \frac{a}{r} \rceil - \lfloor \frac{a}{r} \rfloor) - (a - r\lfloor \frac{a}{r} \rfloor)$ if and only if $x \geq r - (a - r\lfloor \frac{a}{r} \rfloor)$. $\square$

Suppose $\mathcal{C}' \subseteq \mathcal{C}$ is an exact cover for X. Suppose first that r divides $3m + 12q$. Let $S = \{d_1, \ldots, d_m, v_1, \ldots, v_{3q}\} \cup \{c_j | C_j \in \mathcal{C}'\}$. Define $f : V(G) \to \{-1, 0, 1\}$ by

$$f(v) = \begin{cases} 1 & \text{if } v \in S \\ -1 & \text{if } v \in U \\ 0 & \text{otherwise.} \end{cases}$$

Then $f[v] \geq 1$ for all $v \in V(G) - U$. Also, since $|V(G) - U| = 3m + 12q = r\frac{3m+12q}{r} = r\lceil \frac{3m+12q}{r} \rceil = r|V(G)|$, it follows that $|C_f| \geq r|V(G)|$. Furthermore, $f(V(G)) = |S| - |U| = m + 3q + q - (\frac{3m+12q}{r} - (3m + 12q)) = 4m + 16q - \frac{3m+12q}{r} = 4m + 16q - 2\lfloor \frac{3m+12q}{r} \rfloor + \lceil \frac{3m+12q}{r} \rceil$. Hence, f is a function of weight ℓ such that $|C_f| \geq r|V(G)|$. Now suppose that r does not divide $3m + 12q$. Let u be an arbitrary vertex of U and let $S = \{d_1, \ldots, d_m, v_1, \ldots, v_{3q}, u\} \cup \{c_j | C_j \in \mathcal{C}'\}$. Define $f : V(G) \to \{-1, 0, 1\}$ by

$$f(v) = \begin{cases} 1 & \text{if } v \in S \\ -1 & \text{if } v \in U - \{u\} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f[v] \geq 1$ for all $v \in V(G) - (U - \{u\})$. Also, $|V(G) - (U - \{u\})| = 3m + 12q + 1$. Let $a = 3m + 12q$ and $x = 1$. Since $x = 1 \geq r > r - (a - r\lfloor \frac{a}{r} \rfloor)$, we have, by Claim 1, $3m + 12q + 1 \geq r\lceil \frac{3m+12q}{r} \rceil = r|V(G)|$, so that $|C_f| \geq r|V(G)|$. Also, $f(V(G)) = |S| - (|U| - 1) = m + 3q + 1 + q - (\lceil \frac{3m+12q}{r} \rceil - (3m + 12q) - 1) = 4m + 16q - \lceil \frac{3m+12q}{r} \rceil + 2 = 4m + 16q - \lceil \frac{3m+12q}{r} \rceil + 2(\lceil \frac{3m+12q}{r} \rceil - \lfloor \frac{3m+12q}{r} \rfloor) = 4m + 16q - 2\lfloor \frac{3m+12q}{r} \rfloor + \lceil \frac{3m+12q}{r} \rceil = \ell$. Hence, f is a function of weight ℓ such that $|C_f| \geq r|V(G)|$.

We now prove the converse. Among all functions $f : V(G) \to \{-1, 0, 1\}$ for which $f(V(G)) \leq \ell$ and $|C_f| \geq r|V(G)|$, choose one, say f, for which $f(U)$ is a minimum. The minimality of $f(U)$ implies that $f(u) \in \{-1, 1\}$ for all $u \in U$.

Claim 2

$$|C_f| \geq \begin{cases} 3m + 12q & \textit{if } r \textit{ divides } 3m + 12q \\ 3m + 12q + 1 & \textit{if } r \textit{ does not divide } 3m + 12q. \end{cases}$$

Proof. Suppose first that r divides $3m + 12q$. Then $r|V(G)| = r\lceil \frac{3m+12q}{r} \rceil = r\frac{3m+12q}{r} = 3m + 12q$. But $|C_f| \geq r|V(G)|$, so that $|C_f| \geq 3m + 12q$. Suppose now that r does not divide $3m + 12q$. Then Claim 1 implies that $r\lceil \frac{3m+12q}{r} \rceil > 3m + 12q$. But $|C_f| \geq r|V(G)| = r\lceil \frac{3m+12q}{r} \rceil$, so that $|C_f| \geq 3m + 12q + 1$. $\square$

Claim 3 *If u is an endvertex of G, then $u \in C_f$.*

Proof. Suppose, to the contrary, that u is an endvertex of G such that $u \notin C_f$. Let v be the vertex adjacent to u. Since $u \notin C_f$, $f(u)+f(v) \leq 0$, whence $f(u) = f(v) = -1$, or $f(u) = -1$ and $f(v) \geq 0$, or $f(u) = 0$ and $f(v) \leq 0$, or $f(u) = 1$ and $f(v) = -1$.

Case 1. $f(u) = f(v) = -1$. Then $u, v \notin C_f$ and since $|C_f| \geq 3m + 12q$, there exist distinct vertices x and y in U such that $f(x) = f(y) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(w) = \begin{cases} f(w) & \text{if } w \in V(G) - \{u, v, x, y\} \\ 1 & \text{if } w \in \{u, v\} \\ -1 & \text{otherwise.} \end{cases}$$

Then $u, v \in C_g$, $g[w] = f[w]+2$ (where $w \in N(v)-\{u\}$) and $x, y \notin C_g$, so that $|C_g| \geq |C_f|$. Hence, g is a function such that $|C_g| \geq r|V(G)|$ and $g(V(G)) = f(V(G)) \leq \ell$. However, $g(U) < f(U)$, which contradicts our choice of f.

Case 2. $f(u) = -1$ and $f(v) \geq 0$. Then $u \notin C_f$ and since $|C_f| \geq 3m + 12$, there exists $x \in U$ such that $f(x) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(w) = \begin{cases} f(w) & \text{if } w \in V(G) - \{u, x\} \\ 1 & \text{if } w = u \\ -1 & \text{otherwise.} \end{cases}$$

Again g is a function such that $|C_g| \geq r|V(G)|$, $g(V(G)) \leq \ell$, but $g(U) < f(U)$, which contradicts our choice of f.

Case 3. $f(u) = 0$ and $f(v) \leq 0$ or $f(u) = 1$ and $f(v) = -1$. Then $u \notin C_f$ and since $|C_f| \geq 3m + 12$, there exists $x \in U$ such that $f(x) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(w) = \begin{cases} f(w) & \text{if } w \in V(G) - \{v, x\} \\ 1 & \text{if } w = v \\ -1 & \text{otherwise.} \end{cases}$$

Again g is a function such that $|C_g| \geq r|V(G)|$, $g(V(G)) \leq \ell$, but $g(U) < f(U)$, which contradicts our choice of f. $\square$

Claim 4 *If v is a remote vertex of G, then $v \in C_f$.*

Proof. Suppose, to the contrary, that $v \notin C_f$. Let $N(v) = \{u, w\}$ where $\deg(u) = 1$. Then $f(u) + f(v) + f(w) \leq 0$. Claim 3 implies that $f(u) + f(v) \geq 1$, whence $f(w) = -1$. Since $v \notin C_f$, it follows that $f(u) + f(v) \leq 1$, so that $f(u) + f(v) = 1$. Furthermore, $|C_f| \geq 3m + 12q$ implies that there is an $x \in U$ such that $f(x) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in V(G) - \{x, w\} \\ 1 & \text{if } y = w \\ -1 & \text{otherwise.} \end{cases}$$

Then g is a function such that $|C_g| \geq r|V(G)|$, $g(V(G)) \leq \ell$, but $g(U) < f(U)$, which contradicts our choice of f. $\square$

Claim 5 *If w is a vertex (which is not an endvertex) adjacent to a remote vertex then, without loss of generality, we may assume that $f(w) \geq 0$.*

Proof. Suppose $f(w) = -1$. Let v be the remote vertex adjacent to w and let u be the endvertex adjacent to v. Then, since $v \in C_f$ (cf. Claim 4), we have $f(u) = f(v) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in V(G) - \{u, v, w\} \\ 0 & \text{if } y \in \{u, w\} \\ 1 & \text{otherwise.} \end{cases}$$

Then g is a function such that $|C_g| \geq r|V(G)|$, $g(V(G)) = f(V(G))$, $g(U) = f(U)$ and $g(w) \geq 0$. Hence, without loss of generality, we may assume that $f(w) \geq 0$. $\square$

Claim 6 *If v is a remote vertex, then, without loss of generality, we may assume that $f(v) = 1$.*

Proof. Suppose $f(v) \leq 0$. Let u be the endvertex adjacent to v. Then, since $u \in C_f$, $f(v) \geq 0$, whence $f(v) = 0$ and $f(u) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in V(G) - \{u, v\} \\ 1 & \text{if } y = v \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a function such that $|C_g| \geq r|V(G)|$, $g(V(G)) = f(V(G))$, $g(U) = f(U)$ and $g(v) = 1$. Hence, without loss of generality, we may assume that $f(v) = 1$. $\square$

Claim 7 *Without loss of generality we may assume that $f(x_i) \geq 0$ for all $i = 1, \ldots, 3q$.*

Proof. Suppose that, without loss of generality, there exists $i \in \{1, \ldots, 3q\}$ such that $f(x_i) = -1$. Claims 5 and 6 imply that $f(w_i) \geq 0$ and $f(v_i) = 1$.

We show first that $f(w_i) = 1$. For suppose to the contrary that $f(w_i) = 0$. Then $w_i \notin C_f$ and since $|C_f| \geq 3m + 12q$, there exists $x \in U$ such that $f(x) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in V(G) - \{x, w_i\} \\ 1 & \text{if } y = w_i \\ -1 & \text{otherwise.} \end{cases}$$

Then g is a function such that $|C_g| \geq r|V(G)|$, $g(V(G)) \leq \ell$, but $g(U) < f(U)$, which contradicts our choice of f. Hence $f(w_i) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in V(G) - \{x_i, w_i\} \\ 0 & \text{if } y \in \{x_i, w_i\} \end{cases}$$

Then g is a function such that $|C_g| \geq r|V(G)|$, $g(V(G)) = f(V(G))$, $g(U) = f(U)$ and $g(x_i) \geq 0$. Hence, without loss of generality, we may assume that $f(x_i) \geq 0$ for all $i = 1, \ldots, 3q$. $\square$

Claim 8 *$x_i \in C_f$ for all $i = 1, \ldots, 3q$.*

Proof. Suppose, to the contrary, that there exists $i \in \{1, \ldots, 3q\}$ such that $x_i \notin C_f$. Then, by Claims 5 and 7, no vertex of $N[x_i]$ is assigned a -1 by f, so that every vertex of $N[x_i]$ is assigned a 0 by f. Since $x_i \notin C_f$ and $|C_f| \geq 3m + 12q$, there exists $x \in U$ such that $f(x) = 1$. Define $g : V(G) \to \{-1, 0, 1\}$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in V(G) - \{x_i, x\} \\ 1 & \text{if } y = x_i \\ -1 & \text{otherwise.} \end{cases}$$

Then g is a function such that $|C_g| \geq r|V(G)|$, $g(V(G)) \leq \ell$, but $g(U) < f(U)$, which contradicts our choice of f. □

Claims 5, 6 and 7 imply that $w_i \in C_f$ for all $i = 1, \ldots, 3q$ and that $c_j \in C_f$ for $j = 1, \ldots, m$. This, together with Claims 3 and 4, show that $V(G) - U \subseteq C_f$. Let $R = \{c_1, \ldots, c_m\}$, $S = \{x_1, \ldots, x_{3q}\}$, and $T = \{w_1, \ldots, w_{3q}\}$. Let $a = |R \cap P_f|$, $s = |S \cap P_f|$ and $t = |T \cap P_f|$. Since $V(G) - U \subseteq C_f$, Claim 2 implies that $|U \cap C_f| \geq \lceil \frac{3m+12q}{r} \rceil - \lfloor \frac{3m+12q}{r} \rfloor$. Since $f(v) \geq 0$ for all $v \in V(G) - U$ and all endvertices of G are covered, $f(V(G)) \geq 3q + m + a + s + t + \lceil \frac{3m+12q}{r} \rceil - \lfloor \frac{3m+12q}{r} \rfloor - (|U| - \lceil \frac{3m+12q}{r} \rceil + \lfloor \frac{3m+12q}{r} \rfloor) = 3q + m + a + s + t - |U| + 2(\lceil \frac{3m+12q}{r} \rceil - \lfloor \frac{3m+12q}{r} \rfloor) = 3q + m + a + s + t - (\lceil \frac{3m+12}{r} \rceil - (3m + 12q)) + 2(\lceil \frac{3m+12q}{r} \rceil - \lfloor \frac{3m+12q}{r} \rfloor) = 15q + 4m + (a + s + t) - 2\lfloor \frac{3m+12q}{r} \rfloor + \lceil \frac{3m+12q}{r} \rceil$. But $f(V(G)) \leq \ell = 4m + 16q - 2\lfloor \frac{3m+12q}{r} \rfloor + \lceil \frac{3m+12q}{r} \rceil$, so that $a + s + t \leq q$, i.e. $a \leq q - (s + t)$. Hence, at most $3q - 3(s + t)$ vertices of S are adjacent to a vertex of $R \cap P_f$, s vertices in S are assigned a 1 by f and t vertices of S are adjacent to a vertex of $T \cap P_f$. Hence, at most $3q - 2(s + t)$ vertices of S are either adjacent to a vertex of $(R \cup T) \cap P_f$ or assigned a 1 by f. If $s + t > 0$, then there is a vertex in S, say x, such that $f[x] = 0$, contradicting Claim 8. Hence, $s + t = 0$ and $a \leq q$. Since $x_i \in C_f$ for all $i = 1, \ldots, 3q$, $a = q$. It now follows that $\mathcal{C}' = \{C_j | f(c_j) = 1\}$ is an exact cover for X. ■

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