# Some New Localization Theorems on Hamiltonian Properties

F. Tian* and B. Wei†

Institute of Systems Science, Academia Sinica,
Beijing 100080, China

**Abstract**

Let $G$ be a 2-connected graph with $n \geq 3$ vertices such that for any two vertices $u, v$ at distance two in an induced subgraph $K_{1,3}$ or $P_4$ of $G$, the inequality $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - s$ holds for all $w \in N(u) \cap N(v)$. We prove that (i) if $s = 1$ and $|N(u) \cap N(v)| \geq 2$, then $G$ is hamiltonian or $K_{p,p+1} \subseteq G \subseteq K_p + \overline{K}_{p+1}$; (ii) if $s = 0$, then $G$ is either pancyclic, or bipartite graph. This generalizes two localization theorems known before.

## 1. Introduction

In this paper, we consider only simple finite graphs. Our notations and terminology follow Bondy and Murty[3]. For a graph $G$, let $V$ and $E$ denote its vertex set and edge set, respectively. Denote by $d(u,v)$ the distance between $u$ and $v$. $K_{1,3}$ is a graph with 4 vertices in which 3 vertices have degree 1 and the other has degree 3. $P_4$ is a path with 4 vertices. Let $C$ be a longest cycle of $G$ with a fixed cyclic orientation. For $x \in V(C)$, let $x^+$ be the successor and $x^-$ be the predecessor of $x$ in the chosen direction on $C$. A graph $G$ is pancyclic, if for any integer $i, 3 \leq i \leq n$ $G$ has a cycle of length $i$.

The following results are known.

In [4], Hasratian and Khachatrian proved the following theorem:

**Theorem 1.** Let $G$ be a connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|$ for any triple of vertices $u, v, w$ with $d(u,v) = 2$ and $w \in N(u) \cap N(v)$, then $G$ is hamiltonian.

Recently, Theorem 1 was generalized by the following two theorems in [1] and [2]:

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**Theorem 2[2].** Let $G$ be a connected graph with $n \geq 3$ vertices. If for any two vertices $u, v$ with $d(u, v) = 2$ the following conditions hold:

(i) $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - 1$ for all $w \in N(u) \cap N(v)$,

(ii) $|N(u) \cap N(v)| \geq 2$,

then $G$ is hamiltonian or $K_{p,p+1} \subseteq G \subseteq K_p + \overline{K}_{p+1}$, where $n = 2p + 1, p \geq 2$ and $+$ is the join operation.

**Theorem 3[1].** Under the conditions of Theorem 1, $G$ is pancyclic.

In this paper, we obtain the following theorems.

**Theorem 4.** Let $G$ be a 2-connected graph with $n \geq 3$ vertices. If for any two vertices $u, v$ at distance two in an induced subgraph $K_{1,3}$ or $P_4$ of $G$ the following conditions hold:

(i) $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - 1$ for all $w \in N(u) \cap N(v)$,

(ii) $|N(u) \cap N(v)| \geq 2$,

then $G$ is hamiltonian or $K_{p,p+1} \subseteq G \subseteq K_p + \overline{K}_{p+1}$.

**Theorem 5.** Let $G$ be 2-connected graph with $n \geq 3$ vertices. If for any two vertices $u, v$ at distance two in an induced subgraph $K_{1,3}$ or $P_4$ of $G$, the inequality $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|$ holds for any $w \in N(u) \cap N(v)$, then $G$ is either pancyclic or $G$ is a bipartite graph.

Consider the graph $G_1$ obtained from $K_{n-3}$ and $\{x, y, z\}$ by adding an edge set $\{xy, yz, yu, yv, xu, zv\}$, where $\{u, v\} \subseteq V(K_{n-3})$. Obviously, $G_1$ satisfies the conditions of Theorem 4 and Theorem 5, but does not satisfy the conditions of Theorems 1–3, because $|N(x) \cap N(z)| = 1$. Notice that $G$ is 2-connected under the conditions of Theorems 2–3. Therefore, Theorem 4 and Theorem 5 generalize Theorem 2 and Theorem 3, respectively. Also we have the following consequence:

**Corollary 1.** Let $G$ be a 2-connected graph. If $G$ has neither $K_{1,3}$ nor $P_4$ as induced subgraph, then $G$ is pancyclic unless $G$ is a cycle with four vertices.

Notice that if $G$ is $K_{1,3}$-free, then for any $u, v \in V$ with $d(u, v) = 2$, we have $d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| - 1$ for any $w \in N(u) \cap N(v)$. Thus we have

**Corollary 2.** Let $G$ be a 2-connected, $K_{1,3}$-free graph. If for any two vertices $u, v$ at distance two in an induced subgraph $P_4$ of $G$, $|N(u) \cap N(v)| \geq 2$, then $G$ is hamiltonian.

Corollary 2 generalizes a result of Shi [5].

**Corollary 3[5].** Let $G$ be a 2-connected, $K_{1,3}$-free graph. If for any pair of vertices $u, v$ at distance two in $G$, $|N(u) \cap N(v)| \geq 2$, then $G$ is hamiltonian.

## 2. The Proof of Theorem 4

By contradiction, let $G$ be a nonhamiltonian graph that satisfies the conditions of Theorem 4. Clearly, $G$ contains a cycle, since $G$ is 2-connected. Take $C$ a longest cycle with a fixed cyclic orientation. Set $R = G \setminus C$, then $R \neq \emptyset$. Since $G$ is 2-connected, there exists some $v$ in $R$ such that $N(v) \cap V(C) \neq \emptyset$. Choose a vertex $v$

in $R$ such that $|N(v) \cap V(C)| = \max\{|N(v') \cap V(C)| : v' \in R\}$. Let $N(v) \cap V(C) = \{w_1, \cdots, w_t\}$ $(t \geq 1)$. If $t = 1$, since $G$ is 2-connected, there is a path connecting $w_1$ to a vertex, say $y$, of $V(C) \setminus \{w_1\}$ with all internal vertices in $R$. Choose such a shortest path $P' = w_1 v_1 \cdots v_k y$, where $v_i \in R$ for $1 \leq i \leq k$. Since $C$ is a longest cycle of $G$ and $t = 1$, we have $k \geq 2$ and $w_1^+, w_1, v_1, v_2$ are in an induced subgraph $P_4$ of $G$. Thus $|N(w_1^+) \cap N(v_1)| \geq 2$. By the maximality of $C$, $N(w_1^+) \cap N(v_1) \subseteq V(C)$, so that $|N(v_1) \cap V(C)| \geq 2 > t$, a contradiction. Hence $t \geq 2$. Since $C$ is a longest cycle, it is easy to show that for any $1 \leq i < j \leq t$, $w_i^+ w_j^+ \notin E$, $w_i^- w_j^- \notin E$.

Let $C[x, y)$ denote the subpath of $C$ from $x$ to $y$ (in the chosen direction). For $C[x^+, y]$ we also write $C(x, y]$ and similarly, $C[x, y) = C[x, y^-]$. Now, set $C_i = C[w_i, w_{i+1}), i < t$ and $C_t = C[w_t, w_1)$. Suppose $w_i^+ w_i^- \in E$, then $w_{i+1}^- w_i \notin E$. Choose $u_i \in V(C_i)$ such that for any $u \in V(C(w_i, u_i]), uw_i \in E$ but $u_i^+ w_i \notin E$. If $w_i^+ w_i^- \notin E$, set $u_i = w_i^+$. Let $U = \{u_1, u_2, \cdots, u_t\}$ and $W = \{w_1, w_2, \cdots, w_t\}$. Since $C$ is a longest cycle, it is easy to check that for any $1 \leq i \leq t$, $\{v, u_i\}$ is in an induced subgraph $K_{1,3}$ or $P_4$ of $G$ with $d(v, u_i) = 2$ and $U \cup \{v\}$ is an independent set of $G$. Hence $|N(v) \cap N(u_i)| \geq 2$ for all $i$.

If for some $i$, $1 \leq i \leq t$, $N(v) \cap N(u_i) \cap V(G \setminus C) \neq \emptyset$, then we can get a cycle longer than $C$. Hence we may assume that for each $1 \leq i \leq t$, $N(v) \cap N(u_i) \cap V(G \setminus C) = \emptyset$. Thus $N(v) \cap N(u_i) \subseteq W$ for all $1 \leq i \leq t$. Since $\{v, u_i\}$ is in an induced subgraph $K_{1,3}$ or $P_4$ of $G$, $d(v, u_i) = 2$ and $w_i \in N(v) \cap N(u_i)$, we have $d(v) + d(u_i) \geq |N(v) \cup N(u_i) \cup N(w_i)| - 1$ by the hypothesis of Theorem 4. Thus for any $1 \leq i \leq t$, $|N(v) \cap N(u_i)| \geq |N(v) \cup N(u_i) \cup N(w_i)| - |N(v) \cup N(u_i)| - 1 = |N(w_i) \setminus (N(v) \cup N(u_i))| - 1$.

If $|C_{i-1}| \neq 2$ and $w_i^+ w_i^- \notin E$, we have $\{v, w_i^-\} \cup N_U(w_i) \subseteq N(w_i) \setminus (N(v) \cup N(u_i))$. If $|C_{i-1}| = 2$ or $w_i^+ w_i^- \in E$, we have $\{v\} \cup N_U(w_i) \subseteq N(w_i) \setminus (N(v) \cup N(u_i))$. Hence we obtain

$|N_U(w_i)| + 1 + q_i \leq |N(w_i) \setminus (N(v) \cup N(u_i))| \leq |N(v) \cap N(u_i)| + 1 \leq |N_W(u_i)| + 1$ for any $i, 1 \leq i \leq t$, where $q_i = 1$, if $|C_{i-1}| \geq 3$ and $w_i^+ w_i^- \notin E$; otherwise, $q_i = 0$. Therefore,

$$\Sigma_{i=1}^t (|N_U(w_i)| + q_i) \leq \Sigma_{i=1}^t |N_W(u_i)|. \qquad (1)$$

Note that both $\Sigma_{i=1}^k |N_U(w_i)|$ and $\Sigma_{i=1}^k |N_W(u_i)|$ represent the number of edges with one end in $U$ and the other in $W$. From the inequality (1), we obtain that for all $i, 1 \leq i \leq t$, $q_i = 0$.

If there exists some $i$, say $i = s$, such that $|C_s| \geq 3$, then $w_1^+ w_1^- \in E$ by $q_1 = 0$. Since $C$ is a longest cycle of $G$, $|C_1| \geq 3$ and $w_2^+ w_2^- \in E$. For the same reason, we can get for any $i, s-1 \geq i \geq 2$, $|C_i| \geq 3$ and $w_{i+1}^+ w_{i+1}^- \in E$, in turn. Because $|N(v) \cap N(u_i)| \geq 2$, there exists some $j, 2 \leq j \leq s$ such that $w_j u_1 \in E$. Thus we can get a cycle $C' = w_j u_1 u_1^+ \cdots w_j^- w_j^+ \cdots w_1^- w_1^+ \cdots u_1^- w_1 v w_j$ and $|C|' > |C|$, a contradiction.

Hence $|C_i| = 2$ for all $i, 1 \leq i \leq t$. Since $C$ is a longest cycle, there is no path joining two vertices of $U \cup \{v\}$ with all internal vertices in $V(G \setminus C)$. When there exists a vertex $v' (\neq v) \in V(G \setminus C)$ such that $vv' \in E$ or $u_i v' \in E$ for some $1 \leq i \leq t$, since $G$ is 2-connected and $|C_i| = 2$ for all $i$, we can easily get a cycle which is longer than $C$, a contradiction. Thus when $V(G \setminus C) \neq \{v\}$, then for any $v' (\neq v) \in V(G \setminus C)$

we have $N_C(v') \subseteq W$. Hence there exists some $j$ such that $\{v, v'\} \cup N_U(w_j) \subseteq N(w_j) \setminus (N(v) \cup N(u_j))$ and $\{v\} \cup N_U(w_i) \subseteq N(w_i) \setminus (N(v) \cup N(u_i))$ for any $i \neq j$. Similar to the proof of inequality (1), we obtain $\Sigma_{i=1}^{t}|N_U(w_i)| + 1 \leq \Sigma_{i=1}^{t}|N_W(u_i)|$, which is impossible since both $\Sigma_{i=1}^{k}|N_U(w_i)|$ and $\Sigma_{i=1}^{k}|N_W(u_i)|$ represent the number of edges with one end in $U$ and the other in $W$. Hence $V(G \setminus C) = \{v\}$.

Notice that for any $i$ with $1 \leq i \leq t$ we can get a cycle $C' = w_i v w_{i+1} u_{i+1} \cdots w_i$ such that $|C'| = |C|$, $u_i \in V(G \setminus C')$ and $N(u_i) \cap V(C') \neq \emptyset$. Using the same arguments as before, we can derive that $N(u_i) = W$ for any $1 \leq i \leq t$. Hence $K_{p,p+1} \subseteq G \subseteq \overline{K}_{p+1} + K_p$ and $n = 2p + 1$.

Therefore, the proof of Theorem 4 is complete.

## 3. The Proof of Theorem 5

Notice that for any two distinct vertices $u, v$ with $d(u, v) = 2$, if $|N(u)| + |N(v)| \geq |N(u) \cup N(v) \cup N(w)|$, where $w \in N(u) \cap N(v)$, then $|N(u) \cap N(v)| \geq |N(w) \setminus (N(u) \cup N(v))| \geq 2$, since $\{u, v\} \subseteq (N(w) \setminus (N(u) \cup N(v)))$. Thus by Theorem 4, $G$ has a hamiltonian cycle $C$, since otherwise $G$ is a supergraph of $K_{p,p+1}$ and therefore does not satisfy the condition of Theorem 5. If $G$ has no $C_3$, then choose $v \in V(G)$ such that $d(v) = \max\{d(u) : u \in V(G)\}$. When $d(v) = 2$, then $|C| = 4$, that is, $G$ is bipartite. When $d(v) \geq 3$, then $v^-, v, v^+$ are contained in an induced subgraph $K_{1,3}$ of $G$. Since $d(v^-) + d(v^+) \geq |N(v^-) \cup N(v^+) \cup N(v)|$ and $N(v) \cap (N(v^-) \cup N(v^+)) = \emptyset$, we can easily get that $G$ is bipartite.

Therefore, in the rest of this section, we may assume that $G$ contains at least one triangle. For an integer $i \geq 3$, we define $C_i \triangle C_{i+1} = x_1 x_2 \cdots x_{i+1} x_1$ to indicate a cycle with $x_{j-1} x_{j+1} \in E$ for some $1 \leq j \leq i+1$. First we claim that $G$ has a $C_4 \triangle C_5$.

Let $C_3 = x_1 x_2 x_3 x_1$ be a triangle in $G$. By contradiction, assume that there is no $C_4 \triangle C_5$ in $G$. Since $G$ is 2- connected, we may assume that there exist $u \neq v \in V(G \setminus C_3)$ such that $x_1 u \in E, x_3 v \in E$. Then $uv \notin E$, since otherwise we have a $C_4 \triangle C_5$ in $G$, a contradiction. If $u, x_1, x_3, v$ are in an induced subgraph $P_4$ of $G$, then $|N(u) \cap N(x_3)| \geq 2$ and $|N(v) \cap N(x_1)| \geq 2$. When there exists some $w \in V(G) \setminus \{x_1, x_2, x_3, u, v\}$ such that $w \in N(u) \cap N(x_3)$ or $w \in N(v) \cap N(x_1)$, then we can get a $C_4 \triangle C_5$ in $G$, contrary to the assumption. When $(V(G) \setminus \{x_1, x_2, x_3, u, v\}) \cap N(u) \cap N(x_3) = \emptyset$ and $(V(G) \setminus \{x_1, x_2, x_3, u, v\}) \cap N(v) \cap N(x_1) = \emptyset$, then $x_2 u \in E$ and $x_2 v \in E$. Thus we can also get a $C_4 \triangle C_5$, contrary to the assumption.

If $u, x_1, x_3, v$ are not contained in an induced subgraph $P_4$ of $G$, then $ux_3 \in E$ or $vx_1 \in E$, say $ux_3 \in E$. For the same reason, $x_2 v \notin E$ and $(V(G) \setminus \{x_1, x_2, x_3, u, v\}) \cap N(u) \cap N(v) = \emptyset$. Thus $x_2 u \in E$, since otherwise, $u, v, x_2$ are contained in an induced subgraph $K_{1,3}$ of $G$ and $d(u) + d(v) \leq |N(u) \cup N(v) \cup N(x_3)| - 1$, a contradiction. Since $G$ is 2-connected, we may assume that there exists some $w \in V(G) \setminus \{x_1, x_2, x_3, u, v\}$ such that $uw \in E$. By the assumption, $w, u, x_3, v$ are contained in an induced subgraph $P_4$ of $G$ and $N(v) \cap \{x_1, x_2, u, w\}) = \emptyset$. Since $|N(u) \cap N(v)| \geq 2$, there exists some $z \in V(G) \setminus \{x_1, x_2, x_3, w\}$ such that $zu \in E$ and $zv \in E$. Thus we can get a $C_4 \triangle C_5$ in $G$, contrary to the assumption. The final contradiction shows that there must be a $C_4 \triangle C_5$ in $G$.

Now, we shall prove that if $G$ has a $C_i \triangle C_{i+1}$, then $G$ has either a $C_{i+1} \triangle C_{i+2}$ or a $C_{i+2}\Delta C_{i+3}$ for any $i, n-3 \geq i \geq 4$. By contradiction, assume that there exists some $i, 4 \leq i \leq n-3$ such that there is neither $C_{i+1}\Delta C_{i+2}$ nor $C_{i+2}\Delta C_{i+3}$ in $G$. We Choose $v \in V(G \setminus C_{i+1})$ such that $d_{C_{i+1}}(v) = \max\{d_{C_{i+1}}(y) : y \in V(G \setminus C_{i+1})\}$. Let $C_i\Delta C_{i+1} = x_1x_2\cdots x_{i+1}x_1$ and $W = N(v) \cap V(C_{i+1}) = \{w_1, w_2, \cdots, w_t\}$ in order around $C_{i+1}$. Thus we may assume that for any $v' \in V(G \setminus C_{i+1})$ and $x \in V(C_{i+1})$, $\{xv', v'x^+\} \not\subseteq E$, since otherwise, we can get a $C_{i+1}\Delta C_{i+2}$ in $G$, contrary to the assumption. Let $u_j = w_j^+$ for $1 \leq j \leq t$ and $U = \{u_1, u_2, \cdots, u_t\}$. We distinquish the following three cases:

**Case 1.** $t \geq 3$.

**Case 1.1.** For all $1 \leq j \leq t$, $w_j^+w_j^- \in E$.

Then $w_t^+w_1 \notin E$ by the assumption. Choose $u \in V(C(w_t, w_1^-))$ such that $uw_1 \notin E$ but for any $u' \in V(C(u, w_1)$, $u'w_1 \in E$. Since $|N(u^+) \cap N(v)| \geq 2$, there exists some $z \in V(G \setminus C_{i+1})$ or $z \in W \setminus \{w\}$ such that $z \in N(u^+) \cap N(v)$. If $z \in V(G \setminus C_{i+1})$, then we can get a $C_{i+2}\Delta C_{i+3}$ in $G$. If $z \in W$ then we can get a $C_{i+1}\Delta C_{i+2}$ by $t \geq 3$ in $G$, both are contrary to the assumption.

**Case 1.2.** For all $1 \leq j \leq t$, $w_j^+w_j^- \notin E$.

Without loss of generality, we may assume that $x_1x_3 \in E$. Thus $x_2v \notin E$.

If $x_1v \notin E$, then $x_2, x_3 \notin U$. By the assumption, $U \cup \{v\}$ is an independent set of $G$ and $N(u_j) \cap N(v) \subseteq W$ for any $j, 1 \leq j \leq t$. Since $d(v, u_j) = 2$ and $w_j \in N(u_j) \cap N(v)$, we have $d(v) + d(u_j) \geq |N(v) \cap N(u_j) \cap N(w_j)|$. Thus for any $j, 1 \leq j \leq t$,
$|N(v) \cap N(u_j)| \geq |N(v) \cap N(u_j) \cap N(w_j)| - |N(v) \cup N(u_j)| = |N(w_j) \setminus (N(v) \cup N(u_j))|$.
Since $\{v\} \cup N_U(w_j) \subseteq N(w_j) \setminus (N(v) \cup N(u_j))$, we obtain
$|N_U(w_j)| + 1 \leq |N(w_j) \setminus (N(v) \cup N(u_j))| \leq |N(v) \cap N(u_j)| = |N_W(u_j)|$
for any $j, 1 \leq j \leq t$.

Therefore, $\Sigma_{j=1}^t |N_V(w_j)| + t \leq \Sigma_{j=1}^t |N_W(u_j)|$, a contradiction.

If $x_1v \in E$, that is, $w_1 = x_1$, then set $W' = W \setminus \{w_1\}$ and $U' = U \setminus \{u_1\}$. By the assumption, $u_2w_2^- \notin E$, $U' \cup \{v\}$ is an independent set of $G$ and $N(u_j) \cap N(v) \subseteq W$ for each $j, 2 \leq j \leq t$. For the same reason as above, we can get that for any $j, 2 \leq j \leq t$, $|N(v) \cap N(u_j)| \geq |N(w_j) \setminus (N(v) \cup N(u_j))|$. Since $\{v, w_2^-\} \cup N_{U'}(w_2) \subseteq N(w_2) \setminus (N(v) \cup N(u_2))$, we obtain
$|N_{U'}(w_2)| + 2 \leq |N(w_2) \setminus (N(v) \cup N(u_2))| \leq |N(v) \cap N(u_2)| \leq |N_W(u_2)|$
and for any $3 \leq j \leq t$ we have
$|N_{U'}(w_j)| + 1 \leq |N(w_j) \setminus (N(v) \cup N(u_j))| \leq |N(v) \cap N(u_j)| \leq |N_W(u_j)|$.

Therefore, $\Sigma_{j=2}^t |N_{U'}(w_j) + t \leq \Sigma_{j=2}^t |N_W(u_j)| \leq \Sigma_{j=2}^t |N_{W'}(u_j)| + (t-1)$, a contradiction.

**Case 1.3.** There exists some $j, 1 \leq j \leq t$ such that $w_j^+w_j^- \notin E$, denote by $W'$ the set of all such vertices of $W$.

Then by the preceding proof, $W \setminus W' \neq \emptyset$. By the assumption, for all $w \in W'$, $N(v) \cap N(w) \cap V(G \setminus C_{i+1}) = \emptyset$. When $|W'| \leq t-2$, then by the assumption, $U' = \{u^+ : u \in W'\}$ is an independent set of $G$ and $N(u) \cap N(v) \in W'$ for any $u \in U'$. Using the same method as before, we can get a contradiction. When $|W'| = t-1$, say $x_1 = w_1 \in W \setminus W'$, that is, $x_2x_{i+1} \in E$. Set $U' = U \setminus \{u_1\}$. Then by the assumption, $N(u_j) \cap N(v) \subseteq W$ for any $j, 2 \leq j \leq t$. Since $u_2w_2^- \notin E$ and $w_2^- \notin U' \cup N(v)$, using the same method as before, we can get a contradiction.

**Case 2.** $t = 2$, that is, $W = \{w_1, w_2\}$ and $U = \{u_1, u_2\}$.

If $w_j^+ w_j^- \notin E$ for $j = 1$ or $j = 2$, then $|N(v) \cap N(u_j)| \geq |N(w_j) \setminus (N(u_j) \cup N(v))| \geq 3$ and $|N(v) \cap N(w_j^-)| \geq |N(w_j) \setminus (N(w_j^-) \cup N(v))| \geq 3$. Thus $N(v) \cap N(u_j) \cap V(G \setminus C_{i+1}) \neq \emptyset$ and $N(v) \cap N(w_j^-) \cap V(G \setminus C_{i+1}) \neq \emptyset$. Since there exists some $q$, $1 \leq q \leq i+1$ such that $x_{q-1}x_{q+1} \in E$ and $\{x_{q-1}, x_{q+1}\} \neq \{w_j^-, w_j^+\}$, we can get a $C_{i+2}\Delta C_{i+3}$ in $G$, contrary to the assumption.

If $w_j^+ w_j^- \in E$ for $j = 1, 2$, then by the assumption, for $j = 1, 2$ $N(w_j^+) \cap N(v) \subseteq W$ and $N(w_j^-) \cap N(v) \subseteq W$. Set $y_j = w_j^-$ for $j = 1, 2$. Let $C_1$ be the subpath of $C_{i+1}$ from $w_1$ to $y_2$ and $C_2$ be the subpath of $C_{i+1}$ from $w_2$ to $y_1$. Without loss of generality, we may assume $|C_2| \geq |C_1|$. By the assumption, $|C_1| \geq 4$, that is, $|C_{i+1}| \geq 8$.

**Case 2.1.** $u_j^+ w_j \in E$, $j = 1$ or $j = 2$. Say $j = 1$.

If $y_2^- w_1 \in E$, then we can get a $C_{i+1}\Delta C_{i+2} = w_2 y_2 u_2 u_2^+ \cdots y_1 u_1 u_1^+ \cdots y_2^- w_1 v w_2$ in $G$, contrary to the assumption. If $y_2^- w_1 \notin E$, choose $z \in V(C_1)$ such that $z' w_1 \in E$ for any $z' \in C_{i+1}(w_1, z]$ but $z^+ w_1 \notin E$. Then by the assumption, $\{v, w_1, z, z^+\}$ is contained in an induced $P_4$ of $G$ and $N(v) \cap N(z) \in W$. Since $d(v, z) = 2$, $w_2 z \in E$ by the hypothesis of Theorem 5. Thus we can get a $C_{i+1}\Delta C_{i+2} = w_1 z^- \cdots u_1 y_1 y_1^- \cdots u_2 y_2 y_2^- \cdots z w_2 v w_1$, contrary to the assumption.

**Case 2.2.** $u_j^+ w_j \notin E$ for $j = 1, 2$. For the same reason, we may assume that for $j = 1, 2$, $y_j^- w_j \notin E$.

In this subcase, for $j = 1, 2$, $\{v, w_j, u_j, u_j^+\}$ is contained in an induced subgraph $P_4$ of $G$. Thus by the assumption and the hypothesis of Theorem 5, we have $w_1 u_2 \in E$, $w_2 u_1 \in E$ and similarly, $y_1 w_2 \in E$ and $w_1 y_2 \in E$. Consequently, for $j = 1, 2$, $N(w_j) \setminus \{v, u_j\} \subseteq N(u_j) \cup N(v)$ and $N(w_j) \setminus \{v, y_j\} \subseteq N(y_j) \cup N(v)$. Also by the assumption, for any vertices $x^+, x^-$ in $V(C_{i+1}) \setminus \{w_1, w_2\}$, we have $x^+ x^- \notin E$. Since $|C_{i+1}| \leq n - 2$ and $G$ is 2-connected, there exists some $v' \neq v \in V(G \setminus C_{i+1})$ such that $N(v') \cap V(C_{i+1}) \neq \emptyset$. By the choice of $v$, $d_{C_{i+1}}(v') \leq 2$. Let $x \in V(C_{i+1})$ such that $xv' \in E$.

If $x \in V(C_{i+1}) \setminus W$, then by the assumption and Case 2.1, $x^+ x^- \notin E$. Since $d_{C_{i+1}}(v') \leq 2$ and $|N(x) \setminus (N(v') \cup N(x^+))| \geq 3$, $N(v') \cap N(x^+) \cap V(G \setminus C_{i+1}) \neq \emptyset$ by the hypothesis of Theorem 5. Similarly, $N(v') \cap N(x^-) \cap V(G \setminus C_{i+1}) \neq \emptyset$. Thus we can get a $C_{i+2}\Delta C_{i+3}$ in $G$, contrary to the assumption.

If $x = w_j$ for $j = 1$ or $j = 2$, say $j = 1$, then $v' \in N(u_1) \cup N(v)$. When $u_1 v' \in E$, then we can get a $C_{i+1}\Delta C_{i+2}$ in $G$, contrary to the assumption. When $v' \in N(v)$, then by the assumption and the hypothesis of Theorem 5, $d_{C_{i+1}}(v') = 2$. By the preceding case, we may assume $w_2 v' \in E$. Thus we can get a $C_{i+2}\Delta C_{i+3} = w_1 u_1 u_1^+ \cdots y_2 u_2 \cdots y_1 w_2 v v' w_1$, contrary to the assumption.

**Case 3.** $t = 1$. Then by the choice of $v$, for any $v' \in V(G \setminus C_{i+1})$, $d_{C_{i+1}}(v') \leq 1$.

If there exist some $v \in V(G \setminus C_{i+1})$ and some $x \in V(C_{i+1})$ such that $xv \in E$ and $x^+ x^- \notin E$, then by the assumption, $x, x^+, x^-, v$ are in an induced subgraph $K_{1,3}$ of $G$. Thus by the hypothesis of Theorem 5, $N(v) \cap N(x^+) \cap V(G \setminus C_{i+1}) \neq \emptyset$ and $N(v) \cap N(x^-) \cap V(G \setminus C_{i+1}) \neq \emptyset$, since $t = 1$. Because there exists some $j$, $1 \leq j \leq i+1$ such that $x_{j-1} x_{j+1} \in E$ and $\{x_{j-1}, x_{j+1}\} \neq \{x^-, x^+\}$, we can get a $C_{i+2}\Delta C_{i+3}$ in $G$, contrary to the assumption. Thus for any $v \in V(G \setminus C_{i+1})$ and $x \in N(v) \cap V(C_{i+1})$, $x^+ x^- \in E$. Without loss of generality, let $x_1 \in N(v) \cap V(C_{i+1})$.

Since $G$ is 2-connected, there exists a path $P$ connecting $x_1$ and some vertex of

$V(C_{i+1}) \setminus \{x_1\}$ with internal vertices in $V(G \setminus C_{i+1})$. Let $P$ be such a shortest path and $P = x_1 v_1 \cdots v_k x_j$, where $j \neq 1$ and $v_i \in V(G \setminus C_{i+1})$ for any $i$, $1 \leq i \leq k$. Since $t = 1$, we have $k \geq 2$. If $k \geq 3$, then by the choice of $P$, $\{x_2, x_1, v_1, v_2\}$ is contained in an induced subgraph $P_4$ of $G$. Thus by the hypothesis of Theorem 5, there exists some $v^* \in V(G \setminus C_{i+1}) \setminus \{v_1\}$ such that $v^* \in N(x_2) \cap N(v_1)$, since $t = 1$ and consequently, $P' = x_1 v_1 v^* x_2$ is a path which is shorter than $P$, a contradiction. Hence $k = 2$, that is $|P| = 4$. We may assume $P = x_1 v_1 v_2 x_2$.

By the hypothesis of Theorem 5, there exists $v^* \in (G \setminus C_{i+1}) \setminus \{x_2\}$ such that $v^* \in N(x_3) \cap N(v_2)$. Then $\{x_3, x_2, v_2, v_1\}$ is contained in an induced subgraph $P_4$, since $t = 1$. Since $x_{i+1} x_2 \in E$, we get a $C_{i+2} \triangle C_{i+3}$ in $G$, contrary to the assumption.

Therefore Theorem 5 is true.

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