The Number of Classes of Choice Functions under Permutation Equivalence

Spyros S. Magliveras* and Wandi Wei†

Department of Computer Science and Engineering
University of Nebraska-Lincoln
Lincoln, Nebraska 68588-0115

Abstract

A *choice function* f of an $n-$set X is a function whose domain is the power set $\mathcal{P}(X)$, and whose range is X, such that $f(A) \in A$ for each $A \subseteq X$. If k is a fixed positive integer, $k \leq n$, by a *k-restricted choice function* we mean the restriction of some choice function to the collection of $k-$subsets of X. The symmetric group $\mathcal{S}_X$ acts, by natural extension on the respective collections $\mathcal{C}(X)$ of choice functions, and $\mathcal{C}_k(X)$ of $k-$restricted choice functions of X. In this paper we address the problem of finding the number of orbits in the two actions, and give closed form formulas for the respective numbers.

1 Introduction

A *choice function* f on an $n-$set X is defined to be a function whose domain is the collection of all subsets of X, and whose range is X, such that for each subset A of X, $f(A)$ is an element of A. If k is a fixed positive integer, $k \leq n$, by a *k-restricted choice function* we mean the restriction of some choice function to the collection of $k-$subsets of X. If f and g are two choice functions (or $k-$restricted choice functions) on X, we say that f and g are *permutation-equivalent* (or just *equivalent*) if there exists a permutation σ on the symbols of X which transforms f to g. Even though, as we shall see, it is easy to determine the number of classes under permutation equivalence in the collection of all choice functions, it is less trivial to determine the number of equivalence classes of $k-$restricted choice functions. It is perhaps surprising that solutions to these problems have not been given before, even though their statements have been known for awhile. In this paper we address both of these problems, and give closed form formulas for the respective number of equivalence classes.

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2 Preliminaries

For the most part we use standard notation. For example, if Y and Z are sets, Z^Y denotes the collection of all functions $f : Y \to Z$. We denote the collection of all subsets of Z (i.e. the *power set* of Z) by $\mathcal{P}(Z)$, and the collection of all $k-$subsets of Z, by $\binom{Z}{k}$, for $0 \leq k \leq n$. Throughout, X will be an $n-$set, and $\mathcal{S}_X$, or just $\mathcal{S}_n$ will denote the symmetric group on X. Let $\mathcal{C}(X)$ denote the collection of all *choice functions* on X. Thus, $f \in \mathcal{C}(X)$ if and only if $f \in X^{\mathcal{P}(X)}$ and $f(A) \in A$ for all $A \in \mathcal{P}(X)$. Let $\mathcal{C}_k(X)$ denote the collection of all $f \in X^{\binom{X}{k}}$ such that $f(A) \in A$ for all $A \in \binom{X}{k}$. We call $\mathcal{C}_k(X)$ the *k-restricted choice functions* of X. By straightforward counting we easily see that $\mid \mathcal{C}_k(X) \mid = k^{\binom{n}{k}}$, moreover the total number of (unrestricted) choice functions is $\mid \mathcal{C}(X) \mid = \prod_{k=1}^{n} k^{\binom{n}{k}}$.

Let X be a set, and G a multiplicative group, with its identity element denoted by 1. Recall that a *group action* $G|X$ is a function $exp : X \times G \to X$, $(x, g) \mapsto x^g$, satisfying the two axioms (i) $x^1 = x$, $x \in X$, (ii) $(x^g)^h = x^{gh}$, $x \in X$, $g, h \in G$. The $G-$ *orbit* of $x \in X$ is the set $x^G := \{x^g : g \in G\} \subseteq X$, and the *stabilizer* of $x \in X$ the subgroup $G_x = \{g \in G : x^g = x\}$ of G. It is elementary that $|x^G| = [G : G_x]$. If $G|X$ is a group action and $g \in G$, the set of all elements of X fixed by g, $\{x \in X : x^g = x\}$ is called the *fix* of g, and is denoted by $Fix(g)$. The function $\chi : G \to \mathbb{N}$, $\chi(g) := |Fix(g)|$, is called the *character* of the group action. If $G|X$ is a group action we denote by $\rho = \rho(G|X)$ the number of G-orbits on X. Any action $G|X$ extends naturally to an action G on $\mathcal{P}(X)$ defined by $(A, g) \mapsto A^g$ for $A \subseteq X$, $g \in G$. Here $A^g = \{a^g : a \in A\}$. It easily follows that $|A^g| = |A|$. The $G-$orbit of A is the collection of subsets $A^G = \{A^g | g \in G\}$.

If $G|X$ is a group action, $x \in X$, and $\sigma \in G$, by the $\sigma-$*period* of x, $\lambda_\sigma(x)$, we mean the length of the orbit of x under the cyclic group $\langle\sigma\rangle$. If follows that $\lambda_\sigma(x)$ is the least positive integer j such that $x^{\sigma^j} = x$, and that $\lambda_\sigma(x)$ divides the order of σ. Similarly, if $A \subseteq X$, we define the $\sigma-$*period* of A, $\lambda_\sigma(A)$, to be the length of the orbit $|A^{\langle\sigma\rangle}|$. Note that $\lambda_\sigma(\emptyset) = \lambda_\sigma(X) = 1$, for any $\sigma \in \mathcal{S}_n$.

For a general group action $G|X$, the following facts are easy to see: For $x \in X$ and $g, h \in G$, (i) $G_{x^g} = \{G_x\}^g$, (ii) $Fix(g^h) = \{Fix(g)\}^h$, (iii) $\chi(g^h) = \chi(g)$, (iv) $\rho = \frac{1}{|G|}\sum_{g \in G} \chi(g)$. The last equality providing us with the number of $G-$orbits in X is known as the Cauchy-Frobenius lemma. For the remainder of this paper we let $X = \{1, 2, \ldots, n\}$.

3 Main Results

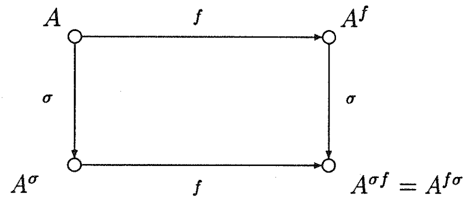
To simplify our discussion, let $\mathcal{C}$ denote either of the sets $\mathcal{C}(X)$ or $\mathcal{C}_k(X)$. If $f \in \mathcal{C}$ it will be useful to write $f = \binom{A_i}{a_i}$, where $a_i = f(A_i) \in A_i$, thus:

$$f = \begin{pmatrix} A_i \\ a_i \end{pmatrix} = \begin{pmatrix} A_1 \; A_2 \ldots A_i \ldots A_r \\ a_1 \; a_2 \ldots \; a_i \; \ldots a_r \end{pmatrix} \tag{3.1}$$

where $r = 2^n$ or $\binom{n}{k}$ accordingly. If $\sigma \in G$, $x \in X$, and $A \subseteq X$, we denote by x^σ, and A^σ the images under σ of x and A respectively. We now extend the action $G|X$ canonically to an action $G|\mathcal{C}$. If $f = \binom{A_i}{a_i} \in \mathcal{C}$, and $\sigma \in G$ we define f^σ by $\binom{A_i}{a_i}^\sigma = \binom{A_i^\sigma}{a_i^\sigma}$. Thus,

$$f^\sigma = \begin{pmatrix} A_i^\sigma \\ a_i^\sigma \end{pmatrix} = \begin{pmatrix} A_1^\sigma\ A_2^\sigma \dots A_i^\sigma \dots A_r^\sigma \\ a_1^\sigma\ a_2^\sigma \dots\ a_i^\sigma \dots a_r^\sigma \end{pmatrix}. \tag{3.2}$$

It is now clear that f^σ is the composition of functions $\sigma^{-1} f \sigma$. Thus, a permutation $\sigma \in G$ fixes $f = \binom{A_i}{a_i} \in \mathcal{C}$ if and only if for all A in the domain of f, the following diagram commutes:



Let $\chi : \mathcal{S}_n \to \mathbb{N}$ denote the character of the action $\mathcal{S}_n \mid \mathcal{C}(X)$, and $\chi_k : \mathcal{S}_n \to \mathbb{N}$ the character of $\mathcal{S}_n \mid \mathcal{C}_k(X)$. The following lemma is evident:

Lemma 3.1 *Let $\sigma \in \mathcal{S}_n$, and $f \in \mathcal{C}$. Then, the following statements are equivalent:*

(i) f is fixed by σ

(ii) For each $A \subseteq X$, $f(A) = a$ if and only if $f(A^\sigma) = a^\sigma$.

We immediately have,

Lemma 3.2 *If σ is a non-identity element of $\mathcal{S}_n$, then σ fixes none of the choice functions in $\mathcal{C}(X)$, i.e. $\chi(\sigma) = 0$.*

Proof: Suppose $f \in \mathcal{C}(X)$, and let $\sigma' = (a_1, \dots, a_t)$, $t \geq 2$, be a non-trivial cycle in the decomposition of σ as the product of disjoint cycles. If $A = \{a_1, \dots, a_t\}$, then, $A^\sigma = A$, while $a_i^\sigma \neq a_i$ for $1 \leq i \leq t$. If $f(A) = a_i$, then $f(A^\sigma) = f(A) = a_i \neq a_i^\sigma$, so by 3.1, f is not fixed by σ. □

Theorem 3.1 *The number of equivalence classes in $\mathcal{C}(X)$ under permutation equivalence is*

$$\rho(n) = \prod_{k=1}^{n} k^{\binom{n}{k}-1}.$$

Proof: Note that a non-identity permutation σ of $\mathcal{S}_n$ fixes $\chi(\sigma) = 0$ functions of $\mathcal{C}(X)$, while the identity permutation fixes all choice functions. Hence, by the Cauchy-Frobenius lemma, the number of $\mathcal{S}_n$−orbits in $\mathcal{C}(X)$ is

$$\rho(n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \chi(\sigma) = \frac{1}{n!} \cdot \prod_{k=1}^{n} k^{\binom{n}{k}} = \prod_{k=1}^{n} k^{\binom{n}{k}-1}.$$

□

If $a \in A \subseteq X$, and $\sigma \in \mathcal{S}_n$, let $\lambda_\sigma(a)$ and $\lambda_\sigma(A)$ be the orbit lengths under $\langle\sigma\rangle$ of the point-orbit $a^{\langle\sigma\rangle}$ and set-orbit $A^{\langle\sigma\rangle}$ respectively. It can be easily seen that, in general, both cases (i) $\lambda_\sigma(a) \leq \lambda_\sigma(A)$, (ii) $\lambda_\sigma(a) > \lambda_\sigma(A)$ are possible. Suppose however, that $f \in \mathcal{C}_k(X)$ is fixed by σ, and $f(A) = a$. Then, it is clear that (ii) can not occur, otherwise the statement (ii) of lemma 3.2 would be violated. Moreover, again under the assumptions that f is fixed by σ and $f(A) = a$, it is easily seen that $\lambda_\sigma(a)$ must divide $\lambda_\sigma(A)$, otherwise condition (ii) of lemma 3.2 would again be violated. Consequently, we have the following

Lemma 3.3 *Let $\sigma \in \mathcal{S}_n$, and suppose that $f \in \mathcal{C}_k(X)$ is fixed by σ. If $\lambda_\sigma(a)$ is the orbit length of $a = f(A)$ under the cyclic group $\langle\sigma\rangle$, and $\lambda_\sigma(A)$ the orbit length of A under $\langle\sigma\rangle$, then, $\lambda_\sigma(a)$ divides $\lambda_\sigma(A)$.*

Let k be a fixed positive integer, $k \leq n$, σ an arbitrary element of $\mathcal{S}_n$, and suppose that $\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_m$ is the factorization of σ as the product of disjoint cycles, where $\sigma_i = (a_{i,1}, \ldots, a_{i,v_i})$. Let $V_i = \{a_{i,1}, \ldots a_{i,v_i}\}$ be the set of elements of X occuring in σ_i. Now, for any $A \in \binom{X}{k}$, $A = A_1 \cup \cdots \cup A_m$, where $A_i = A \cap V_i$. Let $k_i = |A_i|$. Even though some of the A_i may be empty, with corresponding $k_i = 0$, we still call the collection $\{A_1, \ldots, A_m\}$ a partition of A, and $\{k_1, \ldots, k_m\}$ a partition of the natural number k. Since the A_i are disjoint, and $A = A_1 \cup \cdots \cup A_m$, the following lemma is straightforward:

Lemma 3.4 *Under the notation just established, we have:*

(i) $$\lambda_{\sigma_i}(\emptyset) = 1$$

(ii) $$\lambda_\sigma(A_i) = \lambda_{\sigma_i}(A_i)$$

(iii) $$\lambda_\sigma(A) = \text{LCM}\,[\lambda_\sigma(A_1), \ldots, \lambda_\sigma(A_m)].$$

Let $\nu : \mathbb{N} \to \mathbb{N}$ be a *möbius*-like function defined by

$$\nu(\prod_i p_i^{k_i}) = \prod_i (-1)^{k_i} \tag{3.3}$$

where the p_i are primes. Then, we have the following:

Proposition 3.1 *Suppose that $\sigma = (1, 2, \ldots, v)$, and t, k, and v are positive integers such that $t, k \leq v$, and $v = rt$. Let $N(v, k, t)$ denote the number of subsets A of $V = \{1, 2, \ldots, v\}$ such that $|A| = k$, and $\lambda_\sigma(A) = t$. Then,*

$$N(v,k,t) = \sum_{\substack{d \\ r \mid d \mid (k,v)}} \nu(d/r)\binom{v/d}{k/d}. \tag{3.4}$$

Proof: Since $\lambda_\sigma(A) = t$, the stabilizer of A in $\langle\sigma\rangle$ is of order $|\langle\sigma\rangle|/t = v/t = r$. But $\langle\sigma\rangle$ has a unique subgroup of order r, namely the cyclic group generated by $y = \sigma^t$. Now, y is a permutation of type r^t, i.e. it consists of t cycles of length r. Since A is fixed by y, A is the union of some of the t $r-$cycles of y. In particular, $|A| = sr = k$, so that $r|k$, and $s = k/r$. Now, there are $\binom{t}{s} = \binom{v/r}{k/r}$ ways of selecting these s $r-$cycles to form A. However, some of these $k-$sets are also the union of k/d $d-$cycles in the cyclic subgroup of $\langle\sigma\rangle$ order d, where $r \mid d \mid k$. There are $\binom{v/d}{k/d}$ ways of forming the latter $k-$sets, and they fall into orbits of length v/d. Hence, by the inclusion-exclusion principal the result follows. □

Proposition 3.2 *Let σ be a permutation in S_n whose factorization as the product of disjoint cycles is $\sigma = \sigma_1 \cdots \sigma_m$, $|\sigma_i| = v_i$. Then, the number of $k-$restricted choice functions in $C_k(X)$ fixed by σ is*

$$\chi_k(\sigma) = \prod_{\substack{k_1 + \cdots + k_m = k \\ 0 \le k_i \le v_i \\ 1 \le i \le m}} \prod_{\substack{(t_1,\ldots,t_m) \\ t_i \mid v_i \\ k_i = 0 \Rightarrow t_i = 1 \\ 1 \le i \le m}} \Big\{ \sum_{v_i \mid \text{LCM}\,[t_1,\ldots,t_m]} k_i \Big\}^{\{ \prod_{i=1}^m N(v_i,k_i,t_i) \,/\, \text{LCM}\,[t_1,\ldots,t_m] \}}.$$

Proof: For $0 \le i \le m$, let $\sigma_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,v_i})$, and $V_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,v_i}\}$. For a fixed partition of k, $k_1 + \cdots + k_m = k$ with $0 \le k_i \le v_i$, and a fixed $(t_1, \ldots, t_m)$ where the t_i are positive integers, $t_i \mid v_i$, there are $N(v_1,k_1,t_1) \cdots N(v_m,k_m,t_m)$ ways of selecting an m-tuple of fragments $(A_1, A_2, \ldots, A_m)$ such that $A_i \subseteq V_i$, $\cup_i^m A_i = A$, $|A| = k$, and $\lambda_{\sigma_i}(A_i) = t_i$. But, for this fixed $(t_1, \ldots, t_m)$, the collection $\mathcal{F}$ of all $m-$tuples of fragments $(A_1, \ldots, A_m)$, and the collection $\mathcal{F}'$ of corresponding k-sets A, are partitioned into $\langle\sigma\rangle-$orbits all of length LCM $[t_1, \ldots, t_m]$. Now, to form a choice function $f \in C_k(X)$ fixed by σ it is necessary and sufficient to assign a value to a single orbit representative, from each $\langle\sigma\rangle$-orbit of $k-$sets. Moreover, by lemma 3.3, the number of choices is $\sum_{v_i \mid \lambda(A)} k_i$. The result follows. □

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References

[1] Marshall Hall, Jr., Combinatorial Theory, Willey-Interscience in Discrete Math., John Willey & Sons, 1986

[2] H. Wielandt, Finite Permutation Groups, Academic Press, New York, London, 1964.

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