A Nonabelian CI-group

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Abstract

In this note, we prove that the alternating group A_4 is a CI-group and that all disconnected Cayley graphs of A_5 are CI-graphs. As a corollary, we conclude that there are exactly 22 non-isomorphic Cayley graphs of A_4.

Let G be a finite group and set $G^{\#} = G \setminus \{1\}$. For a subset $S \subseteq G^{\#}$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$, the *Cayley graph* is the graph $\mathrm{Cay}(G, S)$ with vertex set G and with x and y adjacent if and only if $yx^{-1} \in S$. For an automorphism σ of G, it easily follows that $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, S^{\sigma})$. The graph $\mathrm{Cay}(G, S)$ is called a *CI-graph* of G if, whenever $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$, there is an element $\sigma \in \mathrm{Aut}(G)$ such that $S^{\sigma} = T$ (CI stands for *Cayley Isomorphism*). A finite group G is called a *CI-group* if all Cayley graphs of G are CI-graphs.

Ádám (1967) conjectured that all finite cyclic groups were CI-groups, and this conjecture was disproved by Elspas and Turner (1970). Since then, a lot of work has been devoted to seeking CI-groups in the literature (see for example [2, 3, 9]). So far, the known CI-groups are the following groups:

(1) $\mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_n, \mathbb{Z}_{2n}$ and $\mathbb{Z}_{4n}$, where n is odd square-free, see [9, 10];

(2) Q_8, $\mathbb{Z}_p^2, \mathbb{Z}_p^3$, where p is a prime, see [3, 5, 11];

(3) D_{2p}, F_{3p} (the Frobenius group of order $3p$), where p is a prime, see [2, 6].

Recently, an explicit list of groups which contains all finite CI-groups was produced by C. E. Praeger and the second author in [8] (also see [6]). Unfortunately, even with this knowledge, it is still a very hard problem to obtain a complete classification of finite CI-groups. By [6], the candidates of indecomposable CI-groups may be divided into three classes, two of them consist of infinite families, and the other contains 9 "sporadic" groups: $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_9$, Q_8,

$Q_8 \rtimes \mathbb{Z}_3$ and $Q_8 \rtimes \mathbb{Z}_9$. With the assistance of computer, B. D. McKay determined cyclic CI-groups of order at most 37, and in particular proved that $\mathbb{Z}_8$ and $\mathbb{Z}_9$ are CI-groups (unpublished). By [11], Q_8 is a CI-group. In this note, we shall prove that $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_3$ ($\cong A_4$) is a CI-group. However, it is not known which of the other sporadic candidates (that is, $\mathbb{Z}_9 \rtimes \mathbb{Z}_2$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_9$, $Q_8 \rtimes \mathbb{Z}_3$ and $Q_8 \rtimes \mathbb{Z}_9$) are CI-groups.

One of interests of studying CI-groups is to classify Cayley graphs of the corresponding groups. As an application of the result that A_4 is a CI-group, we shall give a classification of Cayley graphs of A_4.

For a positive integer m, a group G is called an *m-CI-group* if all Cayley graphs of G of valency at most m are CI-graphs. In [7], it is proved that a nonabelian simple group is a 3-CI-group if and only if it is A_5. However, it is still an open question whether A_5 is a 4-CI-group (see [7]). We shall prove that all disconnected Cayley graphs of A_5 are CI-graphs.

Theorem 1 *The alternating group* A_4 *of order* 12 *is a CI-group.*

Proof. For a positive integer m, a group G is said to have the *m-CI property* if all Cayley graphs of G of valency m are CI-graphs. Clearly, the group G has the m-CI property if and only if G has the $(|G^\#| - m)$-CI property. Therefore, to prove that A_4 is a CI-group, we only need to prove that A_4 has the m-CI property for $m \leq 5$.

Let $G = A_4 \cong \mathbb{Z}_2^2 \rtimes \mathbb{Z}_3$. Then G contains three involutions (elements of order 2): a_1, a_2, a_3, and four subgroups of order 3: $\langle x_i \rangle$ $(i = 1, 2, 3, 4)$. Let $\mathrm{Syl}_3(G)$ be the set of Sylow 3-subgroups of G. It is easily checked that the following properties are true:

(a) $\langle x_i \rangle$ acts (by conjugation) transitively on the set $\{a_1, a_2, a_3\}$;

(b) $\mathbb{Z}_2^2 = \{1, a_1, a_2, a_3\}$ acts (by conjugation) regularly on the set $\mathrm{Syl}_3(G)$;

(c) G acts (by conjugation) 2-transitively on $\mathrm{Syl}_3(G)$.

(If a Cayley graph $\mathrm{Cay}(G, S)$ is a CI-graph, S is called a *CI-subset.*) By (a), we know that G has the 1-CI property. By (a) and (b), it follows that G has the 2-CI property.

Let S be a subset of $G^\#$ of size 3 with $S = S^{-1}$. If S consists of three involutions, then $\langle S \rangle = \mathbb{Z}_2^2$. Noting that G has the unique subgroup $\langle S \rangle$ of order 4, S contains all the involutions of G. For any $T \in G^\#$ such that $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$, we have that $|\langle S \rangle| = |\langle T \rangle|$ and so $S = T$. Thus S is a CI-subset. Let $S = \{x, x^{-1}, a\}$ and $T = \{x', x'^{-1}, a'\}$, where $o(x) = o(x') = 3$ and $o(a) = o(a') = 2$, such that $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$. Since $\langle x \rangle$ is conjugate to $\langle x' \rangle$, there exists an element $y \in G$ such that $\{x, x^{-1}\}^y = \{x', x'^{-1}\}$, so $S^y = \{x', x'^{-1}, a^y\}$. Further, as $\langle x' \rangle$ is transitive on the set of all involutions of G, there exists an integer j such that x'^j maps a^y to a', and so $S^{yx'^j} = \{x', x'^{-1}, a^y\}^{x'^j} = \{x', x'^{-1}, a'\} = T$. Thus S is a CI-subset, and so G has the 3-CI property.

(For $s \in S$, an edge $\{u, v\}$ of $\mathrm{Cay}(G, S)$ is called an *s-edge* if $vu^{-1} = s$.) Let S be a subset of $G^\#$ of size 4 with $S = S^{-1}$, and let

$$R_1 = \{a_1, a_2, x_1, x_1^{-1}\} \text{ and } R_2 = \{x_1, x_1^{-1}, x_2, x_2^{-1}\}.$$

Then an a_1-edge of $\mathrm{Cay}(G, R_1)$ is not an edge of a cycle of length 3, but every edge of $\mathrm{Cay}(G, R_2)$ is an edge of some cycle of length 3. Thus $\mathrm{Cay}(G, R_1) \not\cong \mathrm{Cay}(G, R_2)$. Since $S = S^{-1}$, S contains 0 or 2 involutions. If S contains two involutions, then by properties (a) and (b), S is conjugate to R_1 (arguing as in the previous paragraph); if S does not contain involutions, then it follows from property (c) that S is conjugate to R_2. Therefore, all subsets of $G^\#$ of size 4 are CI-subsets and G has the 4-CI property.

Finally, let S be a subset of $G^\#$ of size 5 with $S = S^{-1}$. Let

$$\begin{aligned} R_1 &= \{a_1, a_2, a_3, x_1, x_1^{-1}\}, \\ R_2 &= \{a_i, x_1, x_1^{-1}, x_2, x_2^{-1}\}, \quad \text{where } \langle x_1^{a_i}\rangle = \langle x_2\rangle, \\ R_3 &= \{a_i, x_1, x_1^{-1}, x_2, x_2^{-1}\}, \quad \text{where } \langle x_1^{a_i}\rangle \neq \langle x_2\rangle. \end{aligned}$$

It is easy to show that $\mathrm{Cay}(G, R_1)$ has a subgraph isomorphic to the complete graph K_4 of order 4 (generated by $\{a_1, a_2, a_3\}$), $\mathrm{Cay}(G, R_2)$ has no subgraph isomorphic to K_4 but every edge of $\mathrm{Cay}(G, R_2)$ is an edge of a cycle of length 3 or 4, and $\mathrm{Cay}(G, R_3)$ has no subgraph isomorphic to K_4 and an a_i-edge of $\mathrm{Cay}(G, R_3)$ is not an edge of a cycle of length less than 5. It follows that $\mathrm{Cay}(G, R_k)$, $k = 1, 2, 3$, are pairwise non-isomorphic. Since $S = S^{-1}$ and G has exactly 3 involutions, S contains 1 or 3 involutions. First suppose that S contains 3 involutions. Then $S = \{a_1, a_2, a_3, x, x^{-1}\}$ for some element x of G of order 3. By property (b), there exists $a \in \mathbb{Z}_2^2$ such that $x^a = x_1$. Thus $S^a = R_1$ and so S is a CI-subset. Next suppose that $S = \{a, x, x^{-1}, y, y^{-1}\}$ such that $\langle x\rangle^a = \langle y\rangle$ where $o(a) = 2$ and $o(x) = o(y) = 3$. By property (c), there exists $g \in G$ such that $\{\langle x\rangle, \langle y\rangle\}^g = \{\langle x_1\rangle, \langle x_2\rangle\}^g$. Thus $S^g = \{a^g, x_1, x_1^{-1}, x_2, x_2^{-1}\}$. Since $\mathbb{Z}_2^2$ acts regularly on $\mathrm{Syl}_3(G)$, we have $a^g = a_i$ so that $S^g = R_2$ and S is a CI-subset. Finally suppose $S = \{a, x, x^{-1}, y, y^{-1}\}$ such that $\langle x\rangle^a \neq \langle y\rangle$ where $o(a) = 2$ and $o(x) = o(y) = 3$. By property (c), S is conjugate to $T := \{a_h, x_1, x_1^{-1}, x_2, x_2^{-1}\}$ for some involution a_h. Assume that $T \neq R$, and consider A_4 as a permutation group on $\{1, 2, 3, 4\}$. Without loss of generality, we may assume $\{x_1, x_1^{-1}, x_2, x_2^{-1}\} = \{(123), (132), (124), (142)\}$. Since $\langle x\rangle^a \neq \langle y\rangle$, $\langle x_1\rangle^{a_h} \neq \langle x_2\rangle$. Therefore, we have $\{a_i, a_h\} = \{(13)(24), (14)(23)\}$, and so we may assume that

$$\begin{aligned} R_3 &= \{(13)(24), (123), (132), (124), (142)\}, \\ T &= \{(14)(23), (123), (132), (124), (142)\}. \end{aligned}$$

Now $T^{(34)} = R_3$, and hence S is conjugate under $\mathrm{Aut}(G)$ to R_3. So S is a CI-subset, and G has the 5-CI property. Therefore, G has the m-CI property for all $m \leq 5$ and so G is a CI-group. □

By Theorem 1 and its proof, it is easy to obtain a complete classification of Cayley graphs of A_4.

Corollary 2 *Up to isomorphism, there are exactly* 22 *Cayley graphs of* A_4.

Proof. By the proof of Theorem 1, it easily follows that there are exactly 1, 1, 2, 2, 2, 3 non-isomorphic Cayley graphs of A_4 of valency 0, 1, 2, 3, 4, 5, respectively. Thus there are exactly 1, 1, 2, 2, 2, 3 non-isomorphic Cayley graphs of A_4 of valency 11, 10, 9, 8, 7, 6, respectively. Therefore, there are exactly 22 non-isomorphic Cayley graphs of A_4. □

Finally, we study isomorphisms of Cayley graphs of A_5.

Theorem 3 *All disconnected Cayley graphs of* A_5 *are CI-graphs.*

Proof. Let $G = A_5$. By [2], the dihedral groups of $2p$ are all CI-groups where p is a prime. Thus both D_6 and D_{10} are CI-groups, and by Theorem 1, A_4 is a CI-group. Therefore, all proper subgroups of G are CI-groups. Let S, T be subsets of G such that $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are disconnected and isomorphic. Then $\langle S\rangle$ and $\langle T\rangle$ are both proper subgroups of G with the same order. It follows that $\langle S\rangle$ is conjugate in $\mathrm{Aut}(G)$ to $\langle T\rangle$. Thus there exists $\sigma \in \mathrm{Aut}(G)$ such that $\langle S\rangle = \langle T\rangle^\sigma$. Let $S' = T^\sigma$. Then $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, S')$ and hence $\mathrm{Cay}(\langle S\rangle, S) \cong \mathrm{Cay}(\langle S\rangle, S')$. Since $\langle S\rangle$ is a CI-group, there is $\alpha \in \mathrm{Aut}(\langle S\rangle)$ such that $S = S'^\alpha$. It is easy to show that all automorphisms of $\langle S\rangle$ can be extended to an automorphism of G. Thus there is $\rho \in \mathrm{Aut}(G)$ such that the restriction of ρ to $\langle S\rangle$ is α. Hence

$$S = S'^\alpha = S'^\rho = (T^\sigma)^\rho = T^{\sigma\rho}.$$

Therefore, $\mathrm{Cay}(G, S)$ is a CI-graph. □

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