MIXED RAMSEY NUMBERS : TOTAL CHROMATIC NUMBER VERSUS STARS

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ABSTRACT :

Given a graph theoretic parameter f, a graph H and a positive integer m, the **mixed ramsey number** f(m,H) is defined as the least positive integer p such that for any graph G of order p either $f(G) \geq m$ or $\overline{G}$ contains H as a subgraph. In this paper we determine the mixed ramsey number $\chi_2(m,K(1,n))$ where χ_2 is the total chromatic number and K(1,n) is the star of order n + 1. This settles a conjecture of Fink.

1. Introduction and Definitions

All graphs considered in this paper are undirected, finite, loopless and have no multiple edges. For the most part our notation and terminology follow that of Chartrand and Lesniak [4]. However, we denote the order and the size of a graph G by $\nu(G)$ and $\varepsilon(G)$ respectively. The degree of a vertex x in a graph G is denoted by $d_G(x)$ and the maximum degree in G is denoted by $\Delta(G)$. If A and B are disjoint subsets of V(G), then G[A,B] denotes the subgraph induced by the edges between A and B. For a subset U of V(G), G[U] denotes the subgraph induced by U. The vertices and edges are referred to as the **elements** of G. A set $V_1 \cup E_1$ (with $V_1 \subseteq V$ and $E_1 \subseteq E$) of elements of G = (V,E) is said to be **independent** if V_1 and E_1 are both independent and no element of V_1 is incident with an element of E_1.

The **total chromatic number** $\chi_2(G)$ of a graph G is the minimum number of colours that can be assigned to the elements of G such that adjacent and incident elements are assigned different colours. It is easy to show that $\chi_2(K_n)$ = n or n + 1 according as n is odd or even. Given a graph parameter f, a positive integer m

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and a graph H, the **mixed ramsey number** $f(m,H)$ is defined as the least positive integer p such that for any graph G of order p either $f(G) \geq m$ or $\overline{G}$ contains H as a subgraph. The concept of mixed ramsey numbers was introduced by Benedict, Chartrand and Lick [2]. In their paper they studied the mixed ramsey numbers concerning chromatic number. Since the publication of their paper a number of authors have considered the mixed ramsey numbers concerning various colouring parameters such as : edge chromatic number (Lesniak et al. [9]); total chromatic number (Fink [6]; Cleves and Jacobson [5]; Zhijian [11]); vertex arboricity (Lesniak-Foster [8]) and path chromatic numbers, vertex linear arboricity and point partition numbers (Achuthan et al. [1]).

The objective of this paper is to determine $\chi_2(m,K(1,n))$ where m and n are positive integers and $K(1,n)$ is the star of order $n + 1$. It is easy to see that $\chi_2(m,K(1,1)) = m - 1$ or m according as m is odd or even. Henceforth we shall assume that $n \geq 2$. We first present a brief survey of the known results. Fink [6] determined the following upper bound for $\chi_2(m,K(1,n))$.

Theorem 1 : If m and n are positive integers with $m \geq 3$, then

$$\chi_2(m,K(1,n)) \leq \begin{cases} m+n-2, & \text{if m is odd and n is even,} \\ m+n-1, & \text{otherwise .} \end{cases}$$

Furthermore, Fink suggested the following

Conjecture (Fink [6]) : For $m \geq 5$ and $n \geq 2$,

$$\chi_2(m,K(1,n)) = \begin{cases} m+n-2, & \text{if m is odd and n is even,} \\ m+n-1, & \text{otherwise.} \end{cases}$$

The conjecture was verified by Fink [6] for some special cases : $m = n$ or $m = n + 3 \equiv 1(\text{mod}2)$. Cleves and Jacobson [5] determined $\chi_2(m,K(1,n))$ for

$3 \le m \le 6$. Zhijian [11] proved that $\chi_2(m,K(1,n)) \ge m + n - 2$ when $m \ge 3$ and $n \ge 1$, thus verifying the conjecture for the case when m is odd and n is even.

In Section 2 we present some constructions of d-regular graphs whose total chromatic number is $d + 1$. Using these constructions we prove the main result in the final section.

2. Constructions of d-regular graphs.

In this section we will present several constructions of d-regular graphs with total chromatic number $d + 1$. The following lemma can easily be proved.

Lemma 1 : Let p and ℓ be integers such that $p = 2\ell$ and $\ell \ge 3$. Let G be isomorphic to $K(\ell,\ell)$ with a perfect matching removed. Then $\chi_2(G) = \ell$.

Lemma 2 : Let p,k and α be integers such that $p = 4k$, $k \ge 2$ and $1 \le \alpha \le 2k - 1$. Let G be a graph of order p where $V(G) = A \cup B$ with $|A| = |B| = 2k$. Let G[A,B] be isomorphic to K(2k,2k) with a perfect matching removed. Also let G[A] and G[B] be isomorphic to the union of α perfect matchings. Then $\chi_2(G) = 2k + \alpha$.

Proof : From Lemma 1 it follows that the elements of G[A,B] can be partitioned into 2k independent sets. Now since G[A] and G[B] are both isomorphic to the union of α perfect matchings, it is possible to partition the edges of $G[A] \cup G[B]$ into α matchings of size 2k each. Thus we have a partition of $V(G) \cup E(G)$ into $2k + \alpha$ independent sets and so $\chi_2(G) \le 2k + \alpha$. Combining this with the fact that G is $(2k + \alpha - 1)$-regular it follows that $\chi_2(G) = 2k + \alpha$.

Lemma 3 : Let $d \ge 3$ be an integer and p an even integer such that $d + 2 \le p \le 2d + 2$. Then there exists a d-regular graph G of order p such that $\chi_2(G) = d + 1$.

Proof : Let $p = 2t$. Then $(d+2)/2 \leq t \leq d + 1$. If t is even, say $t = 2k$, then the graph G of Lemma 2 with $\alpha = d - t + 1$ is the required graph. Thus let t be odd. We now construct a d-regular graph G of order p with $\chi_2(G) = d + 1$. Let $V(G) = A \cup B$, where G[A] and G[B] are both isomorphic to K_t. In addition to these edges G has precisely $d - t + 1$ matchings of size t between A and B. Clearly $\chi_2(G) \geq d + 1$. To prove equality, we will show that the elements of G can be coloured using $d + 1$ colours. We first colour all the vertices of G and the edges of G[A] and G[B] using t colours. It is easy to see that this is possible since $d \leq 2t - 2$ and $\chi_2(K_t) = t$. The remaining edges of G, namely the edges between A and B can now be coloured with $d - t + 1$ new colours. Thus $\chi_2(G) \leq d + 1$. This proves the lemma.

We now state two results that are used to prove Lemma 4.

Theorem 2 (Rees and Wallis [10]) : Let $G \equiv K(m,n)$ with the bipartition (X,Y) where $|X| = m$, $|Y| = n$ and $m \leq n$. Let $Y_1, Y_2, \ldots, Y_n$ be a collection of m-subsets of Y such that every vertex $y \in Y$ is contained in exactly m of the Y_j's. Then the edge set of G can be decomposed into n matchings $M_1, M_2, \ldots, M_n$ where M_i has m edges and is from X to Y_i for $1 \leq i \leq n$.

The following result is a special case of a theorem of Folkman and Fulkerson [7]. A proof of this special case can also be seen in Caccetta and Mardiyono [3].

Theorem 3 (Folkman and Fulkerson [7]) : If G is a graph with ck edges and $c \geq \chi_1(G)$ where $\chi_1(G)$ is the edge chromatic number of G, then the edge set of G admits a decomposition into c matchings each with k edges.

Lemma 4 : Let $d \geq 4$ and p be even and odd integers respectively with $d + 1 \leq p \leq 2d - 1$. There exists a d-regular graph G of order p such that $\chi_2(G) = d + 1$.

Proof : Firstly if $p = d + 1$ then K_p is the required graph. Next let $p = \ell + d$ where ℓ is an odd integer with $3 \le \ell \le d - 1$. Let G be a d-regular graph of order p where $V(G) = A \cup B$ with $|A| = \ell$ and $|B| = d$. Assume that G[B] is isomorphic to $d-\ell$ perfect matchings and $G[A,B] \cong K(\ell,d)$.

Let E be a perfect matching of G[B] and define $H \cong G[B] - E$. Clearly $\varepsilon(H) = d(d-\ell-1)/2$ and $\chi_1(H) = d - \ell - 1$. Using Theorem 3 we can decompose E(H) into d matchings $M_1,M_2,\ldots,M_d$ each of size $(d-\ell-1)/2$. Let Y_i be the set of vertices of H unsaturated by the matching M_i, for $1 \le i \le d$. Clearly $|Y_i| = \ell + 1$, for each i and every vertex of H belongs to exactly $\ell + 1$ of the sets $Y_1,Y_2,\ldots,Y_d$.

Now construct a new graph G' from G by introducing a new vertex ∞ to V(G) and joining it to all the vertices of the set B. By Theorem 2 the edges of $G'[A,B]$ can be decomposed into matchings $M_1',M_2',\ldots,M_d'$ where M_i' is a matching of size $\ell + 1$ from $A \cup \{\infty\}$ to Y_i. Note that each M_i' has exactly one edge of type (∞,x) for some $x \in B$. Now let $T_i = M_i \cup \{M_i' - (\infty,x)\} \cup \{x\}$. Let T_{d+1} be the set consisting of the vertices in A and the edges of E. Clearly $\{T_i : 1 \le i \le d + 1\}$ forms a partition of $V(G) \cup E(G)$ into independent sets. Thus $\chi_2(G) \le d + 1$. Since G is d-regular we have $\chi_2(G) = d + 1$. This proves Lemma 4.

Lemma 5 : Let $d \ge 4$ be an even integer. There exists a d-regular graph G of order $2d + 1$ with $\chi_2(G) = d + 1$.

Proof : Let $d = 2t$, $t \ge 2$. We will divide the proof into two cases depending on the parity of t.

Case i : t is odd, say $t = 2\ell - 1$, $\ell \ge 2$. We now construct a $(4\ell - 2)$ - regular graph G of order $8\ell - 3$ with $\chi_2(G) = 4\ell - 1$. Let $V(G) = X \cup Y \cup Z \cup A$ where $X = \{x_1,x_2,\ldots,x_{2\ell-1}\}$, $Y = \{y_1,y_2,\ldots,y_{2\ell-1}\}$, $Z = \{z_1,z_2,\ldots,z_{2\ell-1}\}$ and $A = \{a_1,a_2,\ldots,a_{2\ell}\}$. Let $G[X] \cong G[Y] \cong G[Z] \cong K_{2\ell-1}$ and $G[A] \cong K_{2\ell}$. Suppose that $\{E_i\}$ is a decomposition of E(G[X]) into 2ℓ-1 matchings each of size ℓ-1 such that x_i is E_i -

unsaturated for each i. Similarly we define $\{F_i\}$ and $\{L_i\}$ corresponding to the graphs $G[Y]$ and $G[Z]$. Also let $\{S_i\}$ be a 1-factorization of $G[A]$. The subgraph $G[X,Y]$ is isomorphic to the union of 2ℓ-2 perfect matchings denoted by $N_1,N_2,\ldots,N_{2\ell-2}$. The subgraph $G[Y,Z]$ is the union of two perfect matchings $N_{2\ell-1}$ and $N_{2\ell}$. We assume without any loss of generality that y_i is not adjacent to x_i or z_i in G.

In addition we introduce precisely 2ℓ-2 matchings $M_1,M_2,\ldots,M_{2\ell-2}$ of size 2ℓ-1 each, between Z and A and two matchings $M_{2\ell-1}$ and $M_{2\ell}$ of size 2ℓ-1 each, between X and A. We assume that a_i is M_i-unsaturated for each i. It is easy to see that G is $(4\ell-2)$-regular and hence $\chi_2(G) \geq 4\ell-1$. To establish equality we consider the following partition of $V(G) \cup E(G)$ into sets T_i, $1 \leq i \leq 2\ell - 1$ and T_i', $1 \leq i \leq 2\ell$.

$$T_i = \{x_i,y_i,z_i\} \cup E_i \cup F_i \cup L_i \cup S_i, \qquad \text{for } 1 \leq i \leq 2\ell - 1,$$

and

$$T_i' = \{a_i\} \cup M_i \cup N_i, \qquad \text{for } 1 \leq i \leq 2\ell.$$

Clearly T_i and T_i' are independent for all i. Thus $\chi_2(G) = 4\ell - 1$. This completes the proof of the lemma in this case.

Case ii : t is even, say $t = 2\ell$, $\ell \geq 1$. We first describe a special decomposition of a $(2\ell - 2)$-regular gaph of order 2ℓ into 2ℓ matchings of size $\ell - 1$ each, which will be used in our construction. Consider the graph $H \cong K_{2\ell}$ - a perfect matching. Let the vertices of H be $1,2,\ldots, 2\ell$ and assume that $(i,i+1)$ is not an edge of H for i $= 1,3,\ldots, 2\ell - 1$. By Theorem 3, it is possible to partition the edges of H into 2ℓ matchings $D_1,D_2,\ldots,D_{2\ell}$ of size $\ell - 1$ each. Let A_i be the set of vertices of H that are D_i-unsaturated. Without any loss of generality let us assume that $i \in A_i$ and $A_i = \{i,i'\}$. Since H is $(2\ell - 2)$-regular it follows that if $i \neq j$ then $i' \neq j'$. We now construct a 4ℓ-regular graph G of order $8\ell+1$ with $\chi_2(G) = 4\ell+1$. Let $V(G) = X \cup Y \cup Z \cup A$ where $X = \{x_1,x_2,\ldots,x_{2\ell}\}$, $Y = \{y_1,y_2,\ldots,y_{2\ell}\}$, $Z = \{z_1,z_2,\ldots,z_{2\ell}\}$ and $A = \{a_1,a_2,\ldots,a_{2\ell+1}\}$ with $G[X] \cong G[Y] \cong G[Z] \cong K_{2\ell}$ and $G[A] \cong K_{2\ell+1}$. Let $\{E_i\}$, $\{F_i\}$ and $\{L_i\}$ be 1-factorizations of $G[X]$, $G[Y]$ and $G[Z]$

respectively. Also let $\{S_i\}$ denote a partition of $E(G[A])$ into $2\ell+1$ matchings of size ℓ each such that a_i is S_i-unsaturated for each i. In addition G has the following edges :

1. Two perfect matchings P_1 and P_2 on $X \cup Y \cup Z$ defined by

   $P_1 = \{(x_i,y_{i+1}),(y_i,z_{i+1}),(z_i,x_{i+1}) : i \text{ odd}, \ 1 \leq i \leq 2\ell-1\}$ and

   $P_2 = \{(x_i,z_{i+1}),(y_i,x_{i+1}),(z_i,y_{i+1}) : i \text{ odd}, \ 1 \leq i \leq 2\ell-1\}$.

2. For each i, $1 \leq i \leq 2\ell$, the collection M_i of edges of the form (y_α,z_β) and (z_α,y_β) where (α,β) is an edge of D_i. Since i and i′ are D_i-unsaturated, it follows that y_i, $y_{i'}$,z_i and $z_{i'}$ are M_i-unsaturated. Also M_i is a matching of size $2\ell-2$ and is disjoint from $P_1 \cup P_2$ and M_j for $j \neq i$.

3. For each i, $1 \leq i \leq 2\ell$, the edges of $J_i - \{(x_i, a_{i_1})\}$ where J_i is a matching of size 2ℓ between X and A which leaves a_i unsaturated and a_{i_1} is the unique neighbour of x_i in J_i. These J_i's are chosen such that $J_i \cap J_j = \phi$ for $i \neq j$.

4. Finally for each i, $1 \leq i \leq 2\ell$, the edges $(a_i,y_{i'})$ and $(a_{i_1},z_{i'})$ between the sets A and $Y \cup Z$.

We now define $N_i = \{J_i - \{(x_i, a_{i_1})\}\} \cup \{(a_i,y_{i'}), (a_{i_1},z_{i'})\}$, for $1 \leq i \leq 2\ell$. Clearly N_i is a matching of size $2\ell + 1$ that saturates every vertex of A, $y_{i'}$,$z_{i'}$ and every vertex of X except x_i.

Clearly G is 4ℓ-regular and hence $\chi_2(G) \geq 4\ell + 1$. To establish equality we partition $V(G) \cup E(G)$ into indpenedent sets T_i, for $1 \leq i \leq 2\ell+1$, and T_i', for $1 \leq i \leq 2\ell$ as follows :

$$T_i = \{a_i\} \cup E_i \cup F_i \cup L_i \cup S_i, \text{ for } 1 \leq i \leq 2\ell - 1,$$

$$T_{2\ell} = \{a_{2\ell}\} \cup S_{2\ell} \cup P_1,$$

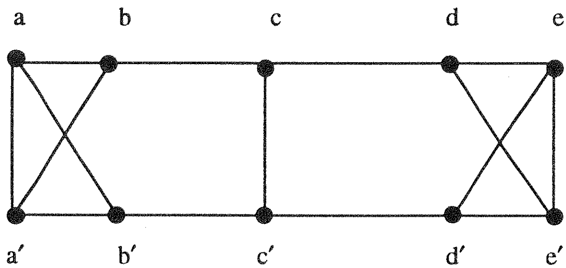
$$T_{2\ell+1} = \{a_{2\ell+1}\} \cup S_{2\ell+1} \cup P_2,$$

and $$T_i' = \{x_i, y_i, z_i\} \cup M_i \cup N_i, \quad \text{for } 1 \leq i \leq 2\ell.$$

It is easy to check that T_i and T_i' are independent for all i. Hence $\chi_2(G) = 4\ell + 1$ and this completes the proof of the lemma.

Lemma 6 : Let $d \geq 3$ be an odd integer and $p = 2d + 4$. There exists a d-regular graph G of order p such that $\chi_2(G) = d + 1$.

Proof : For $d = 3$, the following 3-regular graph (Cleves and Jacobson [5]) of order 10 has the required properties.



Now let $d \geq 5$ and consider the graph G' obtained by taking the disjoint union of two copies H_1 and H_2 of the (d-1)-regular graph of order d+2 along with a d-colouring of the vertices and edges described in the proof of Lemma 4. Now we introduce a matching of size d+2 between the sets $V(H_1)$ and $V(H_2)$ such that no new edge joins vertices that received the same colour in the above d-colouring of G'. Let G be the resulting graph. It is easy to see that $\chi_2(G) = d+1$. This completes the proof of the lemma.

Theorem 4 : Let $d \geq 3$ and p be integers with $p \geq d + 1$. Suppose that p is even and at least as large as $d + 3$ if d is odd. Then there is a d-regular graph G of order p such that $\chi_2(G) = d + 1$.

Proof : Firstly if p is even and $d + 2 \leq p \leq 2d + 2$ the theorem follows from Lemma 3. Next let p be odd and $d + 1 \leq p \leq 2d + 1$. Then d is necessarily even and the theorem follows from Lemmas 4 and 5. Henceforth we will assume that $p \geq 2d + 3$.

Case i : d is odd. In this case p is necessarily even and $p \geq 2d + 4$. Firstly if $p = 2d + 4$, we take G to be the graph of Lemma 6 and hence let $p \geq 2d + 6$. Let us write $p = q(2d + 6) + r$, where $0 \leq r < 2d + 6$. Note that $q \geq 1$ and r is even. We define $G \cong \bigcup_{i=1}^{2q+\delta} G_i$ where $\delta = 0$ or 1 according as $r < d + 3$ or not, $G_{2q+\delta}$ is the graph of Lemma 6 if $r = 2d + 4$ or $d + 1$ and G_i is the graph of Lemma 3 of order λ in all other cases. The value of λ depends on r and i and is given below :

$$\lambda = \begin{cases} r, \text{ if } i = 2q+1 \text{ and } d+3 \leq r \leq 2d+2, \\ d+3+r, \text{ if } i = 2q \text{ and } r \leq d-1, \\ d+3, \text{ if } 1 \leq i \leq 2q+\delta-1. \end{cases}$$

Case ii : d and p are both even. Again $p \geq 2d + 4$ and we write $p = q(2d + 4) + r$, where $0 \leq r < 2d + 4$. Note that $q \geq 1$ and r is even. Let $G \cong \bigcup_{i=1}^{2q+\delta} G_i$ where $\delta = 0$ or 1 according as $r \leq d$ or not and G_i is the graph of Lemma 3 of order λ where λ is given below :

$$\lambda = \begin{cases} d+2, \ 1 \leq i \leq 2q+\delta-1, \\ r, \text{ if } r \geq d+2 \text{ and } i = 2q+1, \\ d+2+r, \text{ if } r \leq d \text{ and } i = 2q. \end{cases}$$

Case iii : d is even and p is odd. We write $p = q(2d + 3) + r$, $0 \leq r \leq 2d + 2$. Note that $q \geq 1$. Firstly let r be even. We define $G \cong \bigcup_{i=1}^{2q+\delta} G_i$ where $\delta = 0$ or 1 according as $r \leq d$ or not, G_i is K_{d+1} for $1 \leq i \leq q$. In all other cases G_i is the graph of Lemma 3 of order λ where λ is defined below :

$$\lambda = \begin{cases} d+2, \text{ for } q+1 \leq i \leq 2q+\delta-1, \\ r, \text{ for } r \geq d+2 \text{ and } i = 2q+1, \\ d+2+r, \text{ for } r \leq d \text{ and } i = 2q. \end{cases}$$

Next let r be odd and $r \neq d + 1$. We define $G \cong \bigcup_{i=1}^{2q+\delta} G_i$ where $\delta = 0$ or 1 according as $r \leq d - 1$ or $r \geq d + 3$, G_i is K_{d+1} for $1 \leq i \leq q$, G_i is the graph of Lemma 3 of order d + 2 for $q + 1 \leq i \leq 2q + \delta - 1$, $G_{2q+\delta}$ is the graph of Lemma 5 of order 2d+1 if $r = 2d + 1$ or $r = d - 1$ and $G_{2q+\delta}$ is the graph of Lemma 4 of order r or d + 2 + r according as $d + 3 \leq r < 2d + 1$ or $1 \leq r < d-1$.

Finally if r = d + 1 then we take G to be the disjoint union of q + 1 copies of K_{d+1} and q copies of the graph of Lemma 3 of order d + 2.

From our construction it is clear that in all the cases G is a d-regular graph of order p. Since the total chromatic number of each component of G is d+1 it follows that $\chi_2(G) = d + 1$. This completes the proof of the theorem.

3. Main Result

We will now verify Fink's [6] conjecture by proving the following :

Theorem 5: For integers $m \geq 5$ and $n \geq 2$,

$$\chi_2(m, K(1,n)) = \begin{cases} m+n-2, & \text{if } m \text{ is odd and } n \text{ is even,} \\ m+n-1, & \text{otherwise.} \end{cases}$$

Proof : To prove **Theorem** 5, it is sufficient to show that there exists a graph G of order p such that $\chi_2(G) \leq m - 1$ and $\Delta(\overline{G}) \leq n-1$, where

$$p = \begin{cases} m+n-3, & \text{if } m \text{ is odd and } n \text{ is even,} \\ m+n-2, & \text{otherwise.} \end{cases}$$

Since $\Delta(G) + 1 \leq \chi_2(G)$, it is sufficient to show the following :

(a) If m is odd and n is even, then there exists a graph G of order m + n - 3 such that $\chi_2(G) \leq m - 1$ and $\delta(G) \geq m - 3$.

(b) Otherwise, there is an (m-2)-regular graph G of order m + n - 2 with $\chi_2(G) = m - 1$.

Firstly if m is odd and n is even then we take G to be an (m - 3)-regular graph of order m + n - 3 such that $\chi_2(G) = m - 2$. Otherwise we take G to be an (m - 2)-regular graph of order m + n - 2 with $\chi_2(G) = m - 1$. The existence of these graphs is guaranteed by **Theorem 4.**

This completes the proof of **Theorem 5.**

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