A Small Embedding For Partial Hexagon Systems

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Dedicated to the memory of Derrick Breach, 1933–1996

1 Introduction

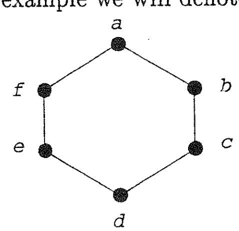
An m-cycle system is a pair (S, C), where C is a collection of edge-disjoint m-cycles which partition the edge set of the complete undirected graph K_n with vertex set S. The *number* $|S| = n$ is called the *order* of the m-cycle system (S, C).

The obvious necessary conditions for the existence of an m-cycle system (S, C) of order n are:

$$\begin{cases} (1)\ n \geq m, \text{ if } n > 1, \\ (2)\ n \text{ is odd, and} \\ (3)\ n(n-1)/2m \text{ is an integer.} \end{cases}$$

Although these necessary conditions are sufficient for all $n \leq 50$ [3], as well as several infinite classes [1, 4, 5, 7, 8, 11], the existence problem is far from settled in general. This will be of no consequence here since we will be considering 6-cycle systems *only*. It is a well-known Folk Theorem that the spectrum for 6-cycle systems (i.e, the set of all n such that a 6-cycle system of order n exists) is *precisely* the set of all $n \equiv 1$ or 9 (mod 12). It is trivial to see that if (S, C) is a 6-cycle system of order n then $|C| = n(n-1)/12$.

A *partial* m-cycle system of order n is a pair (X, P), where P is a collection of edge-disjoint m-cycles of the edge set of K_n with vertex set X. The difference between a partial m-cycle system and an m-cycle system is that the edge-disjoint m-cycles belonging to a partial m-cycle system do not necessarily include all of the edges of K_n. In the following example we will denote the 6-cycle



by any cyclic shift of (a,b,c,d,e,f) or (a,f,e,d,c,b).

Example 1.1. (A partial 6-cycle system of order 11.)

$$\begin{aligned} X &= \{1,2,3,4,5,6,7,8,9,10,11\}, \text{ and} \\ P &= \{(1,2,4,7,3,8),(8,9,11,1,10,2),(4,5,7,10,6,11), \\ &\quad (3,4,6,9,5,10),(2,3,5,8,4,9)\}. \end{aligned}$$

A quite natural question to ask is the following: given a partial m-cycle system (X,P) of order n, is it always possible to decompose $E(K_n)\backslash E(P)$ into m-cycles? ($E(K_n)\backslash E(P)$ is the complement of the edge set of P in the edge set of K_n.) That is, can a partial m-cycle system always be *completed* to an m-cycle system? Example 1.1 shows that this cannot always be done for partial 6-cycle systems, since the completion of Example 1.1 would produce a 6-cycle system of order 11 which is *not* the order of a 6-cycle system.

In general, it is easy to construct partial m-cycle systems which cannot be completed. So, given the fact that a partial m-cycle system cannot necessarily be completed, the following question is immediate. Can a partial m-cycle system always be *embedded* in an m-cycle system?

The partial m-cycle system (X,P) is said to be *embedded* in the m-cycle system (S,C) provided $X \subseteq S$ and $P \subseteq C$.

Example 1.2. (The partial 6-cycle system of order 11 in Example 1.1 is embedded in the 6-cycle system (S,C) of order 13 given below.)

$$\begin{aligned} S &= \{1,2,3,4,5,6,7,8,9,10,11,12,13\}, \text{ and} \\ C &= \{(1,2,4,7,3,8),(8,9,11,1,10,2),(4,5,7,10,6,11),(3,4,6,9,5,10), \\ &\quad (2,3,5,8,4,9),(13,1,3,6,2,7),(12,13,2,5,1,6),(11,12,1,4,13,5), \\ &\quad (10,11,13,3,12,4),(9,10,12,2,11,3),(7,8,10,13,9,1),(6,7,9,12,8,13), \\ &\quad (5,6,8,11,7,14)\}. \end{aligned}$$

In [9, 10] it is shown that a partial m-cycle system of order n can be embedded in an m-cycle system of order $2mn+1$ when n is EVEN and embedded in an m-cycle system of order $m(2n+1)$ when m if ODD.

In [2] the bound for 3-cycle systems (i.e., Steiner triple systems) was reduced from $6n+3$ to *approximately* $4n+1$. In [6] the bound for 4-cycle systems was reduced from $8n+1$ to approximately $2n+\sqrt{n}$. Neither one of these bounds is the best possible. The best possible bound for 3-cycle systems is approximately $2n+1$ and for 4-cycle systems is approximately $n+\sqrt{n}$. "Best possible" means it is always possible to construct for every n *at least* one partial 3-cycle system which cannot be embedded in a 3-cycle system of order less than $2n+1$ *and* a partial 4-cycle system which cannot be embedded in a 4-cycle system of order less than $n+\sqrt{n}$. Whether or not these best possible bounds can be achieved are undoubtedly very difficult problems.

The object of this paper is the reduction of the known bound for embedding partial 6-cycle systems from $12n+1$ to *approximately* $3n$. In particular, we prove that a partial 6-cycle system of order n can always be embedded in a 6-cycle system of order $36k+9$, where $12k \geq n$ is as small as possible. Although this bound is probably not the best, it is certainly a big improvement over $12n+1$!

2 The $36k+9$ Construction

In what follows we will refer to 6-cycles as *hexagons* and 6-cycle systems as *hexagon systems* (this is standard vernacular).

The $36k+9$ Construction. Let Y be a set of size $12k$, Z a set of size 9, and set $S = (Y \times \{1,2,3\}) \cup Z$. Define a collection of hexagons H of the edge set of K_{36k+9} with vertex set S as follows:

(1) Define a hexagon system on $(Y \times \{2\}) \cup Z$ and place these hexagons in H.

(2) For each 2-element subset $\{a,b\}$ of Y, place the hexagon $((a,1),(b,1),(a,2),(b,3),(a,3),(b,2))$ in H.

Let (Y,G,B) be a group divisible design of order $|Y| = 12k$, with groups in G of size 2 and blocks in B of size 3. (Delete any point from a Steiner triple system of order $12k+1$.)

(3) For each block $\{a,b,c\} \in B$ place the hexagon $((a,1),(b,3),(c,1),(a,3),(b,1),(c,3))$ in H.

(4) For each group $\{d,e\} \in G$, place the hexagon $((d,1),(d,2),(d,3),(e,1),(e,2),(e,3))$ in H.

(5) Let ∞_1 and ∞_2 be any two points in Z, and for each group $\{d,e\} \in G$ place the hexagon $(\infty_1,(e,1),(e,3),\infty_2,(d,3),(d,1))$ in H.

(6) Partition the complete bipartite graph with parts $Y \times \{1\}$ and $Z \backslash \{\infty_1\}$ into hexagons and place these hexagons in H. (See Dominique Sotteau [12]).

(7) Partition the complete bipartite graph with parts $Y \times \{3\}$ and $Z \backslash \{\infty_2\}$ into hexagons and place these hexagons in H.

It is straightforward and not difficult to show that (S,H) is a hexagon system of order $36k+9$. (Just count the number of hexagons and show that each edge is in at least one of the hexagons described in (1), (2), (3), (4), (5), (6), or (7).) □

3 Embedding partial hexagon systems

With the $36k+9$ Construction in hand, the embedding follows quite easily.

Theorem 3.1. *A partial hexagon system of order n can be embedded in a hexagon system of order $36k+9$, where $12k \geq n$ is as small as possible.*

Proof. Let (X, P) be a partial hexagon system of order n and $12k \geq n$ as small as possible. Let Y be a set of size $12k$, $X \subseteq Y$, and Z a set of size 9. Let (S, H) be the hexagon system of order $36k + 9$ constructed from Y and Z using the $36k + 9$ Construction. For each hexagon $h = (a, b, c, d, e, f)$ in P denote by $P_1(h)$ the collection of 6 type (2) hexagons $((x, 1), (y, 1), (x, 2), (y, 3), (x, 3), (y, 2))$, where $(x, y) \in \{(a, b), (b, c), (c, d), (d, e), (e, f), (f, a)\}$; and by $P_2(h)$ the collection of 6 hexagons

$$\begin{array}{l} ((a,1),(b,1),(c,1),(d,1),(e,1),(f,1)), \\ ((a,1),(b,2),(c,1),(d,2),(e,1),(f,2)), \\ ((a,2),(b,1),(c,2),(d,1),(e,2),(f,1)), \\ ((a,2),(b,3),(c,2),(d,3),(e,2),(f,3)), \\ ((a,3),(b,2),(c,3),(d,2),(e,3),(f,2)), \text{ and} \\ ((a,3),(b,3),(c,3),(d,3),(e,3),(f,3)). \end{array}$$

The collections of hexagons $P_1(h)$ and $P_2(h)$ are *mutually balanced*; i.e., they contain exactly the same edges. Furthermore, if $h_i \neq h_j \in P$ the edge sets of $P_1(h_i)$ and $P_1(h_j)$ are disjoint. Hence we can replace the collections of hexagons $P_1(h)$, all $h \in H$, with the collections of hexagons $P_2(h)$, all $h \in H$, and the result is a hexagon system of order $36k + 9$. Since $((a, 1), (b, 1), (c, 1), (d, 1), (e, 1), (f, 1))$ and $((a, 3), (b, 3), (c, 3), (d, 3), (e, 3), (f, 3)) \in H$ for each $(a, b, c, d, e, f) \in P$, this new hexagon system contains two disjoint copies of (X, P). □ □

4 Concluding remarks

Although $\approx 3n$ is a big improvement over $12n + 1$, the result in Theorem 3.1 is certainly not the best possible embedding theorem for partial hexagon systems with respect to the size of the containing hexagon system. The author is not quite sure what the best possible embedding "should" be. Lots of work remains to be done on this problem as well as the more general problem of reducing the size of the containing m-cycle systems for the embeddings given in [2, 6, 9, 10].

References

[1] B. Alspach and B. N. Varma, *Decomposing complete graphs into cycles of length $2p^e$*, Annals of Discrete Mathematics, 9 (1980), 155-162.

[2] L. D. Anderson, A. J. W. Hilton, and E. Mendelsohn, *Embedding partial Steiner triple systems*, Proc. London Math. Soc., 41 (1980), 557-576.

[3] E. T. Bell, *Decompositions of K_n into cycles of length at most fifty*, Ars Combinatoria, 40 (1995), 49-58.

[4] J. C. Bermond, C. Huang, and D. Sotteau, *Balanced Cycle and Circuit Designs: Even Cases*, Ars Combinatoria, 5 (1978), 293-318.

[5] D. G. Hoffman, C. C. Lindner, and C. A. Rodger, *On the construction of odd cycle systems*, Journal of Graph Theory, 13, (1989), 417-426.

[6] J. D. Horton, C. C. Lindner, and C. A. Rodger, *A small embedding for partial 4-cycle systems*, J. Combin. Math. Combin. Comput., 5 (1989), 23-26.

[7] B. W. Jackson, *Some cycle decompositions of complete graphs*, J. Combin. Inform. System Sci., 13 (1988), 20-32.

[8] A. Kotzig, *On the decomposition of complete graphs into 4k-gons*, Mat.-Fyz. Časopis Sloven. Akad. Vied., 15 (1965), 229-233.

[9] C. C. Lindner and C. A. Rodger, *A partial* $m = (2k+1)$*-cycle system of order* n *can be embedded in an* m*-cycle system of order* $(2n+1)m$, Discrete Math., 117 (1993), 151-159.

[10] C. C. Lindner, C. A. Rodger, and D. R. Stinson, *Embedding cycle systems of even length*, J. Combin. Math. Combin. Comput., 3 (1988), 65-69.

[11] A. Rosa, *On cyclic decompositions of the complete graph into* $(4m+2)$*-gons*, Mat.-Fyz. Časopis Sloven. Akad. Vied., 16 (1966), 349-353.

[12] D. Sotteau, *Decompositions of* $K_{m,n}(K^*_{m,n})$ *into cycles (circuits) of length* $2k$, J. Combin. Th. (B), 30 (1981), 75-81.

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