Minimum Coverings of $K_n$ with Hexagons

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Abstract
The edge set of $K_n$ cannot be decomposed into edge-disjoint hexagons (or 6-cycles) when $n \not\equiv 1$ or 9 (mod 12). We discuss adding edges to the edge set of $K_n$ so that the resulting graph can be decomposed into edge-disjoint hexagons. This paper gives the solution to this minimum covering of $K_n$ with hexagons problem.

1 Introduction

A hexagon system is a pair $(S, H)$ where $H$ is a collection of edge-disjoint hexagons which partition the edge set of the complete undirected graph $K_n$ with vertex set $S$. The number $|S| = n$ is called the order of the hexagon system, and it is easily seen that $|H| = n(n - 1)/12$. In what follows we will denote the hexagon

by any cyclic shift of $(a, b, c, d, e, f)$ or $(a, f, e, d, c, b)$.

Example 1.1. (hexagon systems of orders 9 and 13)
$S_1 = \{1,2,3,4,5,6,7,8,9\}$; $H_1 = \{(1,2,3,6,7,8), (3,4,5,6,8,9), (1,3,7,4,6,9), (2,4,1,5,3,8), (2,9,4,8,5,7), (1,6,2,5,9,7)\}$
$S_2 = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$; $H_2 = \{(1,2,4,7,3,8), (13,1,3,6,2,7), (2,3,5,8,4,9), (12,13,2,5,1,6), (11,12,1,4,13,5), (10,11,13,3,12,4), (9,10,12,2,11,3), (8,9,11,1,10,2), (7,8,10,13,9,1), (6,7,9,12,8,13), (5,6,8,11,7,12), (4,5,7,10,6,11), (3,4,6,9,5,10)\}$
It is well-known that the *spectrum* for hexagon systems (i.e., the set of all $n$ for which a hexagon system of order $n$ exists) is precisely the set of all $n \equiv 1$ or $9 \pmod{12}$. (See [1, 5] for example.)

If $n \not\equiv 1$ or $9 \pmod{12}$, there is not a hexagon system of order $n$. Some natural questions arise for such $n$. One such question is: “What is the maximum number of edge-disjoint hexagons that can be removed from the edge set of $K_n$ and what do the remaining edges that are not used in the hexagons look like?” This “maximum packing” problem is settled in [2, 3].

Another question to consider is: “What is the fewest number of edges that need to be added to the edge set of $K_n$ so that the edges of the resulting graph can be decomposed into edge-disjoint hexagons, and what does the collection of added edges look like?” This question will be answered in this paper, but first we need a few definitions. A covering of $K_n$ with hexagons is a pair $(S, C)$ where $C$ is a collection of edge-disjoint hexagons which partition the edges of $K_n \cup P$ where $P \subset E(\lambda K_n)$. The collection of edges belonging to $P$ is called the padding and, as with hexagon systems, $n$ is called the order of the covering. If $|P|$ is as small as possible, the covering is called a minimum covering. A covering is simple if $\lambda = 1$, i.e., the padding $P$ is a simple graph. Since a hexagon system is a decomposition of the edges of $K_n$ with no edges added, it is a minimum covering with padding the empty set. Throughout this paper we will refer to minimum coverings of $K_n$ with hexagons simply as minimum coverings.

## 2 Necessary Conditions for Minimum Coverings

We will begin with necessary conditions for simple minimum coverings, and then expand on these conditions for minimum coverings with $\lambda > 1$.

**Simple Minimum Coverings**

If $n$ is odd, every vertex of $K_n$ has even degree, and since each vertex in a hexagon is incident with two edges in that hexagon, we know that each vertex of the padding must have even degree so that each vertex of $K_n \cup P$ has even degree. As we have noted, if $n \equiv 1$ or $9 \pmod{12}$ a hexagon system of order $n$ exists, and this is a minimum covering with padding the empty set. If $n \equiv 3$ or $7 \pmod{12}$, $6|\binom{n}{2} + 3$, hence the smallest possible padding would have three edges, and each vertex having even degree implies the padding would be a 3-cycle. If $n \equiv 11 \pmod{12}$, $6|\binom{n}{2} + 5$, so the smallest possible padding would have five edges, with each vertex having even degree, and the only such simple graph is a 5-cycle. If $n \equiv 5 \pmod{12}$, $6|\binom{n}{2} + 2$, but there is no simple graph with 2 edges and each vertex having even degree, so the smallest possible simple paddings would each have 8 edges with each vertex having even degree. There are 7 such graphs, as we shall see in the next section.

In order for $K_n \cup P$ to have even degree when $n$ is even, $P$ must be a spanning subgraph of $K_n$ with each vertex having odd degree. If $n \equiv 0$ or $6 \pmod{12}$, $6|\binom{n}{2} + \binom{n}{2}$, hence the smallest possible padding is 1-factor, which is the smallest spanning subgraph of odd degree. For $n \equiv 2, 4, 8$ or $10 \pmod{12}$, $6|\binom{n}{2} + \frac{n}{2} + 4$, so the
smallest possible paddings would each have \( \frac{n}{2} + 4 \) edges. There are several such spanning subgraphs of odd degree! If we have a padding with \( \frac{n}{2} + 4 \) edges, the sum of the degrees of its vertices is \( n + 8 \). Since each vertex must have odd degree, the only possible degree sequences for the paddings are \((9,1,1,\ldots,1),(7,3,1,1,\ldots,1),(5,5,1,1,\ldots,1),(5,3,3,1,1,\ldots,1),(3,3,3,3,1,1,\ldots,1)\).

**Minimum Coverings for \( \lambda > 1 \)**

Allowing \( \lambda > 1 \) will reduce the number of edges in the padding in only one case, namely \( n \equiv 5 \pmod{12} \). As mentioned before, for such \( n \), \( 6|\left(\frac{n}{2}\right) + 2 \). Also, each vertex of the padding must have even degree, so for \( \lambda > 1 \) the padding for a minimum covering of order \( n \equiv 5 \pmod{12} \geq 17 \) is a double-edge.

Also, there are cases for which allowing \( \lambda > 1 \) does not change the possible padding for a minimum covering. For \( n \equiv 0 \) or \( 6 \pmod{12} \) the padding is a 1-factor for all \( \lambda \), and for \( n \equiv 3 \) or \( 7 \pmod{12} \), the padding is a 3-cycle for all \( \lambda \).

For \( n \equiv 11 \pmod{12} \) there are 3 more possible paddings for \( \lambda > 1 \), each having five edges with each vertex having even degree. If \( n \equiv 2, 4, 8 \) or \( 10 \pmod{12} \), allowing \( \lambda > 1 \) gives several more possible paddings in each congruency class, each being an odd degree spanning subgraph of \( \lambda K_n \) with \( \frac{n}{2} + 4 \) edges.

### 3 Small Cases

We begin this section with an example of a minimum covering, and then provide a table with all of the possible paddings for each value of \( n \) and \( \lambda \). The minimum coverings are available from the author on request.

**Example 3.1.** \((K_7,C)\), \( \lambda = 1 \): \( P = \{(1,2), (2,3), (1,3)\} \);

\( C = \{(1,2,3,4,6,7), (1,3,2,5,7,4), (1,2,7,3,5,6), (1,3,6,2,4,5)\} \)

**Table 3.1: Paddings for Minimum Coverings**

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</table>

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4 Minimum Coverings

Before we can give the constructions for minimum coverings, we need a few more definitions. A bipartite hexagon system is a triple $\langle X, Y, C \rangle$ where $C$ is a collection of edge-disjoint hexagons which partition the edge set of the complete undirected bipartite graph with vertex set $X \cup Y$ where $X \cap Y = \emptyset$. If $|X| = x$ and $|Y| = y$ the bipartite hexagon system is said to have order $(x, y)$. We also need the following corollary to Sotteau’s Theorem for our constructions.

**Corollary 4.1.** [6] There exists a bipartite hexagon system of order $(6n, 2m)$ for all $6n \geq 6$ and $2m \geq 4$.

**The n+12 Minimum Covering Construction for n Odd**

Let $(K_n, C_1)$ be a minimum covering of odd order $n \geq 7$ based on $X \cup \{\infty\}$, with padding $P$, and $(K_{13}, H_1)$ a hexagon system of order 13 based on $Y \cup \{\infty\}$. Since $n$ is odd, $n - 1$ is even, implying $|X|$ is even, and since $|Y| = 12$, Corollary 4.1 guarantees the existence of a $BHS(X, Y, C_2)$. Define a collection of hexagons $C$ on $X \cup Y \cup \{\infty\}$ by $C = C_1 \cup C_2 \cup H_1$. Then $(K_{n+12}, C)$ is a minimum covering of order $n + 12$ with padding $P$.

**Theorem 4.2.** If $n \equiv 3$ or $7 \pmod{12} \geq 7$, there exists a minimum covering of $K_n$ with padding $P$ if and only if $P$ is a 3-cycle.

*Proof.* Beginning with minimum coverings of orders 7 and 15, the $n + 12$ Minimum Covering Construction yields a minimum covering of every order $n \equiv 3$ or $7 \pmod{12} \geq 19$. 

**Theorem 4.3.** If $n \equiv 5 \pmod{12} \geq 17$, there exists a simple minimum covering of $K_n$ with padding $P$ if and only if $P$ is one of the paddings given in Table 3.1. For $\lambda > 1$, there exists a minimum covering of $K_n$ with padding $P$ if and only if $P$ is a double-edge.

*Proof.* Beginning with the minimum coverings of order 17, the $n + 12$ Minimum Covering Construction yields minimum coverings of every order $n \equiv 5 \pmod{12} \geq 29$.

**Theorem 4.4.** If $n \equiv 11 \pmod{12}$, there exists a minimum covering of $K_n$ with padding $P$ if and only if $P$ is one of the paddings given in Table 3.1.

*Proof.* Beginning with the minimum coverings of order 11, the $n + 12$ Minimum Covering Construction yields minimum coverings with all possible paddings for admissible $n \geq 23$.

Now we move on to minimum coverings of even order, for which we use a slight modification of the previous construction. Again, we stress the following construction is for even $n$. 

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The $n + 6$ Minimum Covering Construction for $n$ Even

Let $(K_n, C_1)$ be a minimum covering of even order $n \geq 6$ based on $X$, with padding $P_1$, and $(K_6, C_2)$ a minimum covering of order 6 based on $Y$, with padding $P_3$. Since $|X|$ is even and $|Y| = 6$, Corollary 4.1 guarantees the existence of a $BHS(X, Y, C_3)$. Define a collection of hexagons $C$ on $X \cup Y \cup \{\infty\}$ by $C = C_1 \cup C_2 \cup C_3$, and let $P = P_1 \cup P_2$. Then $(K_{n+6}, C)$ is a minimum covering of order $n + 6$ with padding $P$.

**Theorem 4.5.** If $n \equiv 0$ or 6 (mod 12) $\geq 6$, there exists a minimum covering of $K_n$ with padding $P$ if and only if $P$ is a 1-factor.

**Proof.** Beginning with the minimum covering of order 6, the $n+6$ Minimum Covering Construction yields a minimum covering of every order $n \equiv 0$ or 6 (mod 12) $\geq 12$. 

**Theorem 4.6.** If $n \equiv 2$ or 8 (mod 12) $\geq 8$, there exists a minimum covering of $K_n$ with padding $P$ if and only if $P$ is a spanning subgraph of $\lambda K_n$ with $\frac{n}{2} + 4$ edges, with each vertex having odd degree. The paddings for $n = 8$ are given in Table 3.1. The paddings for $n = 14$ are those given in Table 3.1 as well as those paddings for $n = 8$ along with 3 independent edges. The paddings for $n = 20$ are the padding given in Table 3.1 as well as those paddings for minimum coverings of orders 8 and 14 along with the appropriate number of independent edges.

**Proof.** Beginning with the minimum coverings of orders 8, 14, and 20, the $n + 6$ Minimum Covering Construction yields all possible minimum coverings of every order $n \equiv 2$ or 8 (mod 12).

**Theorem 4.7.** If $n \equiv 4$ or 10 (mod 12) $\equiv 10$, there exists a minimum covering of $K_n$ with padding $P$ if and only if $P$ is a spanning subgraph of $\lambda K_n$ with $\frac{n}{2} + 4$ edges, with each vertex having odd degree. The paddings for $n = 10$ are given in Table 3.1. The paddings for $n = 16$ are those given in Table 3.1 as well as those for $n = 10$ along with 3 independent edges.

**Proof.** Beginning with the minimum coverings of orders 10 and 16, the $n+6$ Minimum Covering Construction yields all possible minimum coverings of every order $n \equiv 4$ or 10 (mod 12).

5 Summary

The following table gives a brief summary of the results in this paper.
Table 5.1: Summary of Minimum Coverings

<table>
<thead>
<tr>
<th>$K_n$</th>
<th>$\lambda$</th>
<th>Number of Hexagons</th>
<th>Padding</th>
<th>Paddings Possible</th>
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</thead>
<tbody>
<tr>
<td>$n \equiv 1 \text{ or } 9 \pmod{12}$</td>
<td>all</td>
<td>$\frac{n^2 - n}{12}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$n \equiv 3 \text{ or } 7 \pmod{12}$</td>
<td>all</td>
<td>$\frac{n^2 - n + 6}{12}$</td>
<td>3-cycle</td>
<td>1</td>
</tr>
<tr>
<td>$n \equiv 5 \pmod{12}$</td>
<td>1</td>
<td>$\frac{n^2 - n + 4}{12}$</td>
<td>8 edges, all vertices have even degree</td>
<td>7</td>
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<tr>
<td></td>
<td>$\geq 2$</td>
<td>$\frac{n^2 - n + 16}{12}$</td>
<td>double-edge</td>
<td>1</td>
</tr>
<tr>
<td>$n \equiv 11 \pmod{12}$</td>
<td>all</td>
<td>$\frac{n^2 - n + 10}{12}$</td>
<td>5 edges, all vertices have even degree</td>
<td>4</td>
</tr>
<tr>
<td>$n \equiv 0 \text{ or } 6 \pmod{12}$</td>
<td>all</td>
<td>$\frac{n^2}{12}$</td>
<td>1-factor</td>
<td>1</td>
</tr>
<tr>
<td>$n \equiv 2 \text{ or } 8 \pmod{12}$</td>
<td>all</td>
<td>$\frac{n^2 + 8}{12}$</td>
<td>spanning subgraph of odd degree with $\frac{n}{2} + 4$ edges</td>
<td>51</td>
</tr>
<tr>
<td>$n \equiv 4 \text{ or } 10 \pmod{12}$</td>
<td>all</td>
<td>$\frac{n^2 + 8}{12}$</td>
<td>spanning subgraph of odd degree with $\frac{n}{2} + 4$ edges</td>
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References


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