On isomorphisms of Cayley digraphs on dicyclic groups

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Abstract

In this paper we prove that for any $m \in \{1,2,3\}$, the dicyclic group B_{4n} $(n \neq 2)$ is an m-DCI group if and only if n is odd.

1. Introduction

Definition 1.1. Let G be a finite group and let $S \subseteq G \setminus \{1\}$. We define the *Cayley digraph* $X = X(G,S)$ of G with respect to S by

$$V(X) = G,$$
$$E(X) = \{(g, sg) \mid g \in G,\ s \in S\}.$$

It is well-known that any Cayley digraph X on G is vertex-transitive; X is connected if and only if $\langle S \rangle = G$, and X is undirected if and only if $S^{-1} = S$.

Definition 1.2. Let G be a finite group and let $S \subseteq G\setminus\{1\}$. We call S a *CI-subset* of G, if, for any graph isomorphism $X(G,S) \cong X(G,T)$, where $T \subseteq G \setminus \{1\}$, there exists $\alpha \in \operatorname{Aut} G$ such that $S^\alpha = T$.

Definition 1.3. Let G be a finite group and m a positive integer. We call G an *m-DCI-group* if every subset S of $G \setminus \{1\}$ with $|S| \leq m$ is a CI-set. We call G an *m-CI-group* if every subset S of $G \setminus \{1\}$ with $S^{-1} = S$ and $|S| \leq m$ is a CI-subset.

Necessary and sufficient conditions have been found for abelian groups and dihedral groups to be m-DCI-groups, for $m = 1,2,3$ (see [**1–5,8,10**]). The purpose of this paper is to discuss the same problem for dicyclic groups, B_{4n}. The group B_{4n} is defined by

$$B_{4n} = \langle a, b \mid a^{2n} = 1,\ b^2 = a^n,\ b^{-1}ab = a^{-1}\rangle, \ (n \geq 2).$$

If $n = 2$, B_{4n} is isomorphic to the quaternion group Q_8 of order 8, and Q_8 is a 3-DCI-group, [**9**]. Our main result is

Theorem 1.4. *The finite dicyclic group B_{4n} $(n \neq 2)$ is an m-DCI-group for $m = 1, 2, 3$ if and only if n is odd.*

The notation and terminology used in this paper are standard in general; the reader is referred to [**6,7**] when necessary. Let X and Y be two isomorphic Cayley digraphs on G. We use $I(X,Y)$ to denote the set of all isomorphisms ϕ from X to Y with $1^\phi = 1$. For any $u \in V(X)$ and each positive integer i, we write $X_i(u) = \{v \in V(X) \mid d(u,v) = i\}$, where $d(u,v)$ is the distance from u to v.

2. Preliminary results

In this section we shall prove several lemmas which will be used in the proof of Theorem 1.4.

Lemma 2.1. *Let G be a finite group and S, T subsets of $G \setminus \{1\}$ with $X = X(G,S) \cong X(G,T) = Y$.*

(i) *If there exist $s \in S$, $t \in T$ such that $s^\phi = t$ holds for all $\phi \in I(X,Y)$, then $I(X,Y) \subseteq I(X(G, S \setminus \{s\}), X(G, T \setminus \{t\}))$.*
(ii) *If there exists $x \in G$ such that $x^\phi = x$ holds for all $\phi \in I(X,Y)$, then $I(X,Y) \subseteq I(X(G, xS^2 \setminus \{1\}), X(G, xT^2 \setminus \{1\}))$, where $xS^2 = \{xs_is_j \mid s_i, s_j \in S\}$ and $xT^2 = \{xt_it_j \mid t_i, t_j \in T\}$.*

Proof. We use $R(g)$ to denote the map from G to G defined by $x \to xg$.

(i) For any $g \in G$, since $R(g)\phi R((g^\phi))^{-1}) \in I(X,Y)$, by assumption (i) we obtain that $s^{R(g)\phi R((g^\phi)^{-1})} = t$, so that $(sg)^\phi = tg^\phi$. So, for any $g_1, g_2 \in G$, $g_2 = sg_1$ if and only if $g_2^\phi = (sg_1)^\phi = tg_1^\phi$. That is, $(g_1, g_2) \in E(X(G, S \setminus \{s\}))$ if and only if $(g_1^\phi, g_2^\phi) \in E(X(G, T \setminus \{t\}))$.
(ii) As in (i), with $s = t = x$, we have $(xg)^\phi = xg^\phi$ for any $g \in G$. If $(g_1, g_2) \in E(X(G, xS^2 \setminus \{1\}))$, then there exist $s_i, s_j \in S$ such that $g_2 = xs_is_jg_1$ and $xs_is_j \neq 1$, and so $g_2^\phi = (xs_is_jg_1)^\phi = x(s_is_jg_1)^\phi = xt_kt_\ell g_1^\phi$, where $t_k, t_\ell \in T$, and $xt_kt_\ell \neq 1$. Hence $(g_1^\phi, g_2^\phi) \in E(X(G, xT^2 \setminus \{1\}))$. □

Lemma 2.2. ([**10**])

(i) *For $m = 1$, 2, 3, the finite cyclic group Z_k is an m-DCI group if $4 \nmid k$.*
(ii) *Any finite cyclic group Z_k is a 4-DCI group.*

Lemma 2.3. *For any $\pi \in Aut\langle a\rangle$ and for any integer k, define the mapping $\pi' : B_{4n} \to B_{4n}$ by*

$$(a^ib^j)^{\pi'} = (a^i)^\pi(a^kb)^j, \quad (i = 0, 1, \ldots, 2n-1, \ j = 0, 1).$$

Then $\pi' \in Aut\, B_{4n}$.

A finite group G is said to be *homogeneous* if for any two isomorphic subgroups H and K of G and any isomorphism σ from H to K, σ can be extended to an automorphism α of G. Obviously, the finite cyclic group Z_k is homogeneous.

Lemma 2.4. *The finite dicyclic group B_{4n} is homogeneous if $2 \nmid n$.*

Proof. Let H, K be two isomorphic subgroups of B_{4n} and let σ be an isomorphism from H to K.
Case 1. $H \le \langle a \rangle$. Since $|H| = |K|$ and $4 \nmid |H|$, $K \le \langle a \rangle$. The conclusion follows from Lemma 2.3 and the fact that the cyclic group is homogeneous.
Case 2. $H \cap \langle a \rangle b \ne \emptyset$. Then $K \cap \langle a \rangle b \ne \emptyset$. Since $|H : H \cap \langle a \rangle| = |K : K \cap \langle a \rangle| = 2$, there are i, $j \in \{0, 1, \dots, 2n-1\}$ such that $H = \langle H \cap \langle a \rangle, a^i b \rangle$, $K = \langle K \cap \langle a \rangle, a^j b \rangle$, and $(a^i b)^\sigma = a^j b$. Furthermore, $(H \cap \langle a \rangle)^\sigma = K \cap \langle a \rangle$. Thus there exists $\pi \in \mathrm{Aut}\,\langle a \rangle$ such that $\pi|_{H\cap\langle a \rangle} = \sigma|_{H \cap \langle a \rangle}$. By Lemma 2.3, there exists $\pi' \in \mathrm{Aut}\,B_{4n}$ such that $\pi'|_{\langle a \rangle} = \pi$ and $(a^i b)^{\pi'} = a^j b$. Obviously, π' is an extension of σ to B_{4n}. □

3. Proof of Theorem 1.4

We shall discuss, respectively, the cases $m = 1, 2, 3$. Theorem 1.4 is proved by Lemmas 3.1, 3.2 and 3.4. Throughout this section, B_{4n} is assumed to be a fixed finite dicyclic group.

Lemma 3.1. *B_{4n} $(n \ne 2)$ is a 1-DCI-group if and only if n is odd.*

Proof. "only if". Assume $2 \mid n$. Then $o(a^{\frac{n}{2}}) = 4$, so $X(B_{4n}, \{a^{\frac{n}{2}}\}) \cong X(B_{4n}, \{b\})$. Since $n \ne 2$, $\langle a^{\frac{n}{2}} \rangle$ is a characteristic subgroup of B_{4n}. Thus $\{a^{\frac{n}{2}}\}$ is not a CI-subset of B_{4n}, which contradicts the fact that B_{4n} is a 1-DCI group.
"if". This is trivial by Lemma 2.4. □

Lemma 3.2. *B_{4n} $(n \ne 2)$ is a 2-DCI-group if and only if n is odd.*

Proof. By Lemma 3.1, we need only prove that if $2 \nmid n$, $S \subseteq B_{4n} \setminus \{1\}$ and $|S| = 2$, then S is a CI-subset of B_{4n}.

Assume $X = X(B_{4n}, S) \cong X(B_{4n}, T) = Y$.

Let $S = \{a^n, x\}$, where $x \in B_{4n} \setminus \{1\}$. There are just one directed edge and one undirected edge starting from every vertex of $X(B_{4n}, S)$, so we can assert that $T = \{a^n, y\}$. If we delete all undirected edges from $X(B_{4n}, S)$ and $X(B_{4n}, T)$, we obtain that $X(B_{4n}, \{x\}) \cong X(B_{4n}, \{y\})$. By Lemma 3.1 there exists $\sigma \in \mathrm{Aut}\,B_{4n}$ such that $x^\sigma = y$. Obviously, $S^\sigma = T$. Thus we shall assume that $S \ne \{a^n, x\}$.
Case 1. $|S \cap \langle a \rangle| = 2$. Since $|\langle S \rangle| = |\langle T \rangle|$ and $4 \nmid |\langle S \rangle|$, $|T \cap \langle a \rangle| = 2$. By Lemmas 2.2 and 2.3, S is a CI subset of B_{4n}.
Case 2. $|S \cap \langle a \rangle| = 0$. By Lemma 2.3, without loss of generality, we may assume that $S = \{b, a^i b\}$. Since $|X_1(b) \cap X_1(a^i b)| = |\{a^n\}| = 1$, we can assert that $|T \cap \langle a \rangle| = 0$, and we may also assume that $T = \{b, a^j b\}$. Clearly, $(a^n)^\phi = a^n$ for all $\phi \in I((X, Y)$. By Lemma 2.1(ii), we obtain that $X(B_{4n}, a^n S^2 \setminus \{1\}) \cong X(B_{4n}, a^n T^2 \setminus \{1\})$. Thus $X(\langle a \rangle, \{a^{\pm i}\}) \cong X(\langle a \rangle, \{a^{\pm j}\})$. By Lemma 2.2, there exists $\pi \in \mathrm{Aut}\,\langle a \rangle$ such that $\{a^{\pm i}\}^\pi = \{a^{\pm j}\}$. We apply Lemma 2.3, with $k = 1$ if $(a^i)^\pi = a^j$ or $k = j$ if $(a^i)^\pi = a^{-j}$. In either case, we obtain a map $\pi' \in \mathrm{Aut}\,B_{4n}$ such that $S^{\pi'} = T$.
Case 3. $|S \cap \langle a \rangle| = 1$. From the above analysis we may assume that $S = \{b, a^i\}$ and $T = \{b, a^j\}$ and, since $|\langle S \rangle| = |\langle T \rangle|$, $o(a^i) = o(a^j)$. So there exists $\sigma \in \mathrm{Aut}\,B_{4n}$ such that $S^\sigma = T$. □

Lemma 3.3. ([8]). *If $X(Z_k,S)\cong X(Z_k,T)$, where $S=\{\pm i,\pm j,\pm(i-j)\}$, $T=\{\pm u,\pm v,\pm(u-v)\}$, and $|S|=|T|=6$, then there is an automorphism $\pi\in \operatorname{Aut} Z_k$ such that $S^\pi=T$.*

Remark. [8] requires that k be odd. However, the proof given in [8] does not need this restriction.

Lemma 3.4. *B_{4n} $(n\neq 2)$ is a 3-DCI-group if and only if n is odd.*

Proof. By Lemma 3.2, we need only prove that if $2\nmid n$, $S\subseteq B_{4n}\setminus\{1\}$ and $|S|=3$, then S is a CI-subset of B_{4n}.

Assume $X=X(B_{4n},S)\cong X(B_{4n},T)=Y$.

Let $S=\{a^n,x,y\}$, where $x,y\in B_{4n}\setminus\{1\}$. It is easy to see that $a^n\in T$. Assume that $T=\{a^n,u,v\}$. If $x,y\in\langle a\rangle$, it is easy to see that S is a CI-subset of B_{4n}.

If $x\neq y^{-1}$, it is obvious that $(a^n)^\phi=a^n$ for all $\phi\in I(X,Y)$.

If $x=y^{-1}=a^ib$, since $X_2(1)\cap X_1(1)=\{a^n\}$, $(a^n)^\phi=a^n$ for all $\phi\in I(X,Y)$.

By Lemma 2.1(i), we obtain $X(B_{4n},\{x,y\})\cong X(B_{4n},\{u,v\})$. By Lemma 3.2, there exists $\sigma\in\operatorname{Aut} B_{4n}$ such that $\{x,y\}^\sigma=\{u,v\}$. Obviously, $S^\sigma=T$. Thus we shall suppose that $S\neq\{a^n,x,y\}$ in the following. Now we discuss, respectively, $|S\cap\langle a\rangle|=0$, 1, 2 or 3.

Case 1. $|S\cap\langle a\rangle|=3$. By the same argument as in Case 1 of Lemma 3.2, S is a CI-subset of B_{4n}.

Case 2. $|S\cap\langle a\rangle|=2$. Without loss of generality, we may assume that $S=\{b,a^i,a^j\}$.

(i) First we verify that there exists a fixed $t\in T$ such that $b^\phi=t$, for all $\phi\in I(X,Y)$. This is obvious if $a^i=a^{-j}$. Assume $a^i\neq a^{-j}$. We consider $X_2(1)=\{a^{2i},a^{2j},a^{i+j},a^n,a^ib,a^jb,a^{-i}b,a^{-j}b\}$. Then $7\le|X_2(1)|\le 8$.

If $|X_2(1)|=8$, since $|X_1(a^i)\cap X_1(a^j)|=1$, $|X_1(b)\cap X_1(a^i)|=|X_1(b)\cap X_1(a^j)|=0$.

If $|X_2(1)|=7$, since $X_2(a^i)\cap X_2(a^j)\cap X_1(1)=\{b\}$, $X_2(a^i)\cap X_2(b)\cap X_1(1)=X_2(a^j)\cap X_2(b)\cap X_1(1)=\emptyset$.

(ii) Secondly, by Lemma 2.1(i), we obtain $X(B_{4n},\{b\})\cong X(B_{4n},\{t\})$. Thus, by Lemma 3.2, $|T\cap\langle a\rangle|=2$. We can also assume that $T=\{b,a^u,a^v\}$ and $b^\phi=b$ for all $\phi\in I(X,Y)$. So we obtain $X(\langle a\rangle,\{a^i,a^j\})\cong X(\langle a\rangle,\{a^u,a^v\})$. Hence, by Lemmas 2.2 and 2.3, we obtain that S is a CI-subset of B_{4n}.

Case 3. $|S\cap\langle a\rangle|=0$. Assume $S=\{b,a^ib,a^jb\}$. Since $|X_1(b)\cap X_1(a^ib)\cap X_1(a^jb)|=|\{a^n\}|=1$, we deduce that $|T\cap\langle a\rangle|=0$, and we can assume that $T=\{b,a^ub,a^vb\}$. Clearly, $(a^n)^\phi=a^n$ for all $\phi\in I(X,Y)$. By Lemma 2.1(ii), we obtain $X(B_{4n},\{a^{\pm i},a^{\pm j},a^{\pm(i-j)}\})\cong X(B_{4n},\{a^{\pm u},a^{\pm v},a^{\pm(u-v)}\})$. Hence $X(\langle a\rangle,\{a^{\pm i},a^{\pm j},a^{\pm(i-j)}\})\cong X(\langle a\rangle,\{a^{\pm u},a^{\pm v},a^{\pm(u-v)}\})$. If $|\{a^{\pm i},a^{\pm j},a^{\pm(i-j)}\}|=2$ or 4, by Lemma 2,2 there exists $\pi\in\operatorname{Aut}\langle a\rangle$ such that $\{a^{\pm i},a^{\pm j},a^{\pm(i-j)}\}^\pi=\{a^{\pm u},a^{\pm v},a^{\pm(u-v)}\}$. So the conclusion is immediate.

Now suppose $|\{a^{\pm i},a^{\pm j},a^{\pm(i-j)}\}|=6$. By Lemma 3.3. there exists $\pi\in\operatorname{Aut}\langle a\rangle$ such that $\{a^{\pm i},a^{\pm j},a^{\pm(i-j)}\}^\pi=\{a^{\pm u},a^{\pm v},a^{\pm(u-v)}\}$. Without loss of generality, we may assume $(a^i)^\pi=a^u$. Then $(a^{-i})^\pi=a^{-u}$.

(1) If $(a^j)^\pi=a^{-v}$, then $2u\equiv 0$ or $2v\equiv 0 \pmod{2n}$, which contradicts the

fact that $|\{a^{\pm i}, a^{\pm j}, a^{\pm(i-j)}\}| = 6$.

(2) If $(a^j)^\pi = a^{-(u-v)}$, we can get a sinilar contradiction.

(3) If $(a^j)^\pi = a^v$, it is obvious that S is a CI-subset of B_{4n}.

(4) If $(a^j)^\pi = a^{(u-v)}$, by Lemma 2.3 there exists $\pi' \in \operatorname{Aut} B_{4n}$ such that $b^{\pi'} = a^u b$ and $(a^m)^{\pi'} = [(a^m)^\pi]^{-1}$. So $S^{\pi'} = T$.

Case 4. $|S \cap \langle a \rangle| = 1$. By the above analysis, we get immediately that $|T \cap \langle a \rangle| = 1$. Assume $S = \{b, a^i b, a^j\}$ and $T = \{b, a^u b, a^v\}$ ($j \neq n$, $v \neq n$).

(i) First we verify that $(a^j)^\phi = a^v$ for all $\phi \in I(X, Y)$. This is obvious if $b = (a^i b)^{-1}$. Assume $b \neq (a^i b)^{-1}$. We consider

$$X_2(1) = \{a^{2j}, a^n, a^{i+n}, a^{-i+n}, a^j b, a^{-j} b, a^{i+j} b, a^{i-j} b\}.$$

Then $7 \leq |X_2(1)| \leq 8$.

Suppose $|X_2(1)| = 8$. Since $|X_1(b) \cap X_1(a^1 b)| = 1$, $|X_1(b) \cap X_2(a^j)| = |X_1(a^i b) \cap X_1(a^j)| = 0$.

Suppose $|X_2(1)| = 7$. Then exactly one of the following congruence formulae holds:

$$i + n \equiv 2j, \ -i + n \equiv 2j, \ 1 + 2j \equiv 0 \text{ or } -i + 2j \equiv 0 \pmod{2n}.$$

(1) If $i + n \equiv 2j$ or $-i + n \equiv 2j \pmod{2n}$, then $|X_1(1) \cap X_2(a^j)| = |\{b, a^j\}| = 2$, $|X_1(1) \cap X_2(b)| \leq 1$ and $|X_1(1) \cap X_2(a^i b)| \leq 1$, so the result follows.

(2) If $i+2j \equiv 0$ or $-i+2j \equiv 0 \pmod{2n}$, then $|X_1(1) \cap X_3(1)| = |\{b, a^i b\}|$ and the result follows.

(ii) Secondly, since $(a^n)^\phi = a^n$ for all $\phi \in I(X, Y)$, by Lemma 2.1(i) for all $\phi \in I(X, Y)$ we have $\phi \in I(X(B_{4n}, a^n S'^2 \setminus \{1\}), X(B_{4n}, a^n T'^2 \setminus \{1\}))$, where $S' = \{b, a^i b\}$, $T' = \{b, a^u b\}$, and hence $\phi \in I(X(B_{4n}, \{a^i, a^{-i}\}), X(B_{4n}, \{a^u, a^{-u}\}))$.

On the other hand, from the proof of Lemma 2.1(i), $(a^j g)^\phi = a^v g^\phi$ holds for any $g \in B_{4n}$. So it is easy to show that $X(B_{4n}, \{a^i, a^{-i}, a^j\}) \cong X(B_{4n}, \{a^u, a^{-u}, a^v\})$, and so $X(\langle a \rangle, \{a^i, a^{-i}, a^j\}) \cong X(\langle a \rangle, \{a^u, a^{-u}, a^v\})$. By Lemma 2.2, there exists $\pi \in \operatorname{Aut} \langle a \rangle$ such that $\{a^{\pm i}\}^\pi = \{a^{\pm u}\}$ and $(a^j)^\pi = a^v$. We apply Lemma 2.3, with $k = 1$ if $(a^i)^\pi = a^u$ or $k = u$ if $(a^i)^\pi = a^{-u}$. In either event we obtain a map $\pi' \in \operatorname{Aut} B_{4n}$ such that $S^{\pi'} = T$. □

It is easy to see that B_{4n} is a 1-CI-group for any $n \geq 2$. From the above Lemmas we have the following:

Corollary 3.5. B_{4n} ($n \neq 2$) *is an m-CI-group,* $m = 2, 3$ *if and only if* n *is odd.*

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