Automorphism Groups and Isomorphisms of Cayley Digraphs of Abelian Groups*

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Abstract

Let S be a minimal generating subset of the finite abelian group G. We prove that if the Sylow 2-subgroup of G is cyclic, then S and $S \cup S^{-1}$ are CI-subsets and the corresponding Cayley digraph and graph are normal.

Let G be a finite group and let S be a subset of G not containing the identity element 1. The Cayley digraph $X = \mathrm{Cay}(G, S)$ of G with respect to S is defined by

$$V(X) = G,$$
$$E(X) = \{(g, sg) \mid g \in G, s \in S\}.$$

Obviously we have the following basic facts.

Proposition 1 *Let $X = \mathrm{Cay}(G, S)$ be the Cayley digraph of G with respect to S. Then*

(1) $\mathrm{Aut}(X)$ *contains the right regular representation $R(G)$ of G.*
(2) X *is connected if and only if $G = \langle S \rangle$.*
(3) X *is undirected if and only if $S^{-1} = S$.*

We call a subset S of G a CI-subset, if for any subset T of G with $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$, there is an automorphism α of G such that $S^\alpha = T$. A Cayley digraph $X = \mathrm{Cay}(G, S)$ is called normal if $R(G) \lhd A = \mathrm{Aut}(X)$.

Xu [1, Problem 6] asked the following Question (for part (1), see also [2, Problem 8]).

Question 2 *Let G be a finite group and let S be a minimal generating set of G.*

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(1) *Are* S *and* $S \cup S^{-1}$ *CI-subsets ?*

(2) *Are the corresponding Cayley digraph and graph normal ?*

For cyclic groups, Huang and Meng [3, 4, 5] proved

Proposition 3 *Let* G *be a finite cyclic group and let* S *be a minimal generating set of* G. *Let* $X = \mathrm{Cay}(G, S)$ *and* $\overline{X} = \mathrm{Cay}(G, S \cup S^{-1})$. *Let* σ *be an automorphism of* G *such that* $g^\sigma = g^{-1}$, $\forall g \in G$, *and let* $\Sigma = \langle \sigma \rangle$. *Then* $\mathrm{Aut}(X) = R(G)$ *and* $\mathrm{Aut}(\overline{X}) = R(G)\Sigma$. *The answers to both parts* (1) *and* (2) *in Question* 2 *are positive.*

There is a obvious error in the second assertion of this Proposition. (Let $G = Z_{12} \cong \langle a \rangle$ and $S = \{a^3, a^4\}$. It is easy to check $|\mathrm{Aut}(G, S \cup S^{-1})| = 4$ where $\mathrm{Aut}(G, S \cup S^{-1}) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$, so $\mathrm{Aut}(\overline{X}) \neq R(G)\Sigma$.). However it is true that the answers to both questions (1) and (2) are still positive; we prove this in the Theorem below for a larger family of finite abelian groups than the cyclic groups.

For abelian groups, Li [6] gave an example which shows that the answer to question (1) is negative in general. (This is also true for question (2); if let $G = Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle$ and $S = \{a, ab\}$, then both $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, S \cup S^{-1})$ are not normal.). However, if the group has odd order, then the answer to (1) is positive. Namely, he proved

Proposition 4 (1) *Let* $G = \langle a \rangle \times \langle x \rangle \times \langle e \rangle \cong Z_3 \times Z_4 \times Z_2$ *and let* $S = \{x, xe, ax^2\}$ *and* $T = \{x, xe, ax^2e\}$. *Then* S *is a minimal generating subset of* G *and the Cayley digraph* $\mathrm{Cay}(G, S)$ *is isomorphic to* $\mathrm{Cay}(G, T)$. *However, there is no automorphism of* G *which maps* S *to* T. *In other words,* S *is not a CI-subset.*

(2) *Every minimal generating subset of an abelian group of odd order is a CI-subset.*

Feng and Xu [7] proved that all generating subsets of an abelian group G with the minimum number of generators are CI, that is, the answer to question (1) for minimum generating sets of a finite abelian group is positive. Actually they proved

Proposition 5 *Let* G *be a finite abelian group and let both* S *and* T *be minimal generating subsets of* G *of minimum size. Suppose that* $X = \mathrm{Cay}(G, S)$ *and* $Y = \mathrm{Cay}(G, T)$ *are isomorphic. Then there exists an* $\alpha \in \mathrm{Aut}(G)$ *such that* $S^\alpha = T$.

Let G be Li's example in Proposition 4. Then the Sylow 2-subgroup of G is not cyclic. The main result of this paper is the following Theorem:

Theorem *Let* G *be a finite abelian group such that the Sylow* 2*-subgroup of* G *is cyclic. Let* S *be a minimal generating subset of* G. *We have*

(1) S *and* $S \cup S^{-1}$ *are CI-subsets.*

(2) *The corresponding Cayley digraph and graph are normal.*

As a consequence of this, every minimal generating subset of a cyclic group is CI and every minimal generating subset of an abelian group of odd order is CI. (These are Huang and Meng's and Li's results.)

Proof of Theorem: To prove the theorem, first we need the following.

Fact 1: Let $x_1, x_2 \in S$ and $x_1 \neq x_2$. Then $x_1^2 \neq x_2^2$.

Proof of Fact 1: Assume $x_1^2 = x_2^2$. Let $m = o(x_1)$, the order of x_1. If m is odd then $1 = x_1^m = x_1 \cdot x_1^{m-1} = x_1 \cdot x_2^{m-1}$, a contradiction to the minimality of S. If m is even then $a = x_1^{\frac{m}{2}}$ is the unique involution in G. By $x_1^2 = x_2^2$ we have $a = x_1^{-1}x_2$; so $x_2 = ax_1 = x_1^{\frac{m}{2}}x_1$, a similar contradiction. □

Now we are ready to prove that S is CI. Let σ be an isomorphism from $X = \mathrm{Cay}(G,S)$ to $Y = \mathrm{Cay}(G,T)$ such that $1^\sigma = 1$. Set $S = \{x_1, x_2, \cdots, x_n\}$ and $x_i' = x_i^\sigma \in T$ $(1 \le i \le n)$.

Assume $x_i x_k = x_j x_l$ $(i \ne j)$ where $x_i, x_j, x_k, x_l \in S$. By the minimality of S we have $k = i$ or j. Similarly $l = i$ or j. Since $x_i^2 \ne x_j^2$ (Fact 1), we have $x_k = x_j$ and $x_l = x_i$. Thus the intersection of the out-neighborhoods $X_1(x_i)$ and $X_1(x_j)$ of x_i and x_j in the digraph X is

$$X_1(x_i) \cap X_1(x_j) = \{x_i x_j\} \quad (i \ne j).$$

Since $x_i^\sigma x_j^\sigma \in Y_1(x_i^\sigma) \cap Y_1(x_j^\sigma)$ $(i \ne j)$, we must have $(x_i x_j)^\sigma = x_i^\sigma x_j^\sigma = x_i' x_j'$ $(i \ne j)$, and hence also $(x_i^2)^\sigma = (x_i^\sigma)^2 = (x_i')^2$ $(1 \le i \le n)$. Thus

$$(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})^\sigma = (x_1')^{i_1}(x_2')^{i_2} \cdots (x_n')^{i_n} \qquad (*)$$

where i_1, i_2, $\cdots$, i_n are non-negative integers and $i_1 + i_2 + \cdots + i_n \le 2$.

Assume $x^\sigma = x'$, $(xx_i)^\sigma = x'x_i'$ $(1 \le i \le n)$. The same argument as in the proof of $(*)$ will give

$$(x x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})^\sigma = x'(x_1')^{i_1}(x_2')^{i_2} \cdots (x_n')^{i_n} \qquad (**)$$

where i_1, i_2, $\cdots$, i_n are non-negative integers and $i_1 + i_2 + \cdots + i_n \le 2$.

Now we shall prove that $(*)$ holds for any non-negative integers i_1, i_2, $\cdots$, i_n. We use induction on $i_1 + i_2 + \cdots + i_n$.

Assume that $i_1 + i_2 + \cdots + i_n > 2$. Taking $x = x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ such that $0 \le j_k \le i_k$ $(k = 1, 2, \cdots, n)$ and $j_1 + j_2 + \cdots + j_n = i_1 + i_2 + \cdots + i_n - 2$, the inductive hypothesis will give $x^\sigma = x' = (x_1')^{j_1}(x_2')^{j_2} \cdots (x_n')^{j_n}$ and $(xx_i)^\sigma = x'x_i'$ $(1 \le i \le n)$. Then $(**)$ will give the desired result. This shows that $\sigma \in \mathrm{Aut}(G)$, so S is a CI-subset.

Next we shall prove that $S \cup S^{-1}$ is CI. Let α be an isomorphism from $\overline{X} = \mathrm{Cay}(G, S \cup S^{-1})$ to $\overline{Y} = \mathrm{Cay}(G, \overline{T})$ such that $1^\alpha = 1$. Put $T = S^\alpha$. We still set $S = \{x_1, x_2, \cdots, x_n\}$ and $x_i' = x_i^\alpha \in T$ $(1 \le i \le n)$. In order to prove that $S \cup S^{-1}$ is CI, we need the following Fact 2.

Fact 2: Let $x_i \in S$. Then $(x_i^{-1})^\alpha = (x_i^\alpha)^{-1}$.

Proof of Fact 2: We have two cases.

(1) $o(x_i) \ne 2, 4$.

Suppose $x_i x_j^{\delta_j} = x_i^{-1} x_k^{\delta_k}$ $(\delta_k, \delta_j = \pm 1)$ where $x_j, x_k \in S$. Clearly $\{x_1^{\delta_1}, x_2^{\delta_2}, \cdots, x_n^{\delta_n}\}$ $(\delta_i = \pm 1, i = 1, 2, \cdots, n)$ are minimal generating subsets of G, and hence we have $j = k$. Since $o(x_i) \ne 2, 4$ and for $m \ne n$, $x_m^2 \ne x_n^2, (x_n^{-1})^2$ (Fact 1), we obtain that $x_j^{\delta_j} = x_i^{-1}$ and $x_k^{\delta_k} = x_i$. Thus $\overline{X}_1(x_i) \cap \overline{X}_1(x_i^{-1}) = \{1\}$ where $\overline{X}_1(x_i)$ and $\overline{X}_1(x_i^{-1})$ denote the neighborhoods of x_i and x_i^{-1} in the graph $\overline{X}$. On the other hand if $y \in \overline{T}$ and $y \ne (x_i^\alpha)^{-1}$, then $|\overline{Y}_1(x_i^\alpha) \cap \overline{Y}_1(y)| \ge 2$ (since $1, x_i^\alpha y \in \overline{Y}_1(x_i^\alpha) \cap \overline{Y}_1(y)$). Hence $(x_i^{-1})^\alpha = (x_i^\alpha)^{-1}$.

(2) $o(x_i) = 2$ or 4.

If $o(x_i) = 2$ then for arbitrary $x \in S$, $o(x) \neq 4$ by the minimality of S (for if $o(x_i) = 4$, then $x_i = x^2$). If $o(x_i) = 4$ then similarly for arbitrary $x \in S$, $o(x) \neq 2$. Thus $\forall x \in S\backslash\{x_i, x_i^{-1}\}$, $(x^{-1})^\alpha = (x^\alpha)^{-1}$ by (1). Noting that G has only one involution, we have $(x_i^{-1})^\alpha = (x_i^\alpha)^{-1}$. □

By Fact 2, we have $\overline{T} = T \cup T^{-1}$ and $\langle T \rangle = G$. Thus we can use the method of proving S is CI to prove $S \cup S^{-1}$ is CI.

Assume $x_i x_k^{\delta_k} = x_j x_l^{\delta_l}$ ($\delta_k, \delta_l = \pm 1$, $i \neq j$) where $x_i, x_j, x_k, x_l \in S$. By the minimality of $\{x_1^{\delta_1}, x_2^{\delta_2}, \cdots, x_n^{\delta_n}\}$ ($\delta_i = \pm 1$, $i = 1, 2, \cdots, n$), $k = i$ or j and $l = i$ or j. Since $x_i^2 \neq x_j^2$ (Fact 1), we can easily obtain $\overline{X}_1(x_i) \cap \overline{X}_1(x_j) = \{1, x_i x_j\}$ ($i \neq j$). Since $x_j^\alpha \neq (x_i^\alpha)^{-1}$ (by Fact 2, $(x_i^{-1})^\alpha = (x_i^\alpha)^{-1}$) and $x_i^\alpha x_j^\alpha \in \overline{Y}_1(x_i^\alpha) \cap \overline{Y}_1(x_j^\alpha)$, we have $(x_i x_j)^\alpha = x_i^\alpha x_j^\alpha$ ($i \neq j$).

Similarly $(x_i x_j^{-1})^\alpha = x_i^\alpha (x_j^{-1})^\alpha = x_i^\alpha (x_j^\alpha)^{-1}$ ($i \neq j$). Thus $(x_i^2)^\alpha = (x_i^\alpha)^2$ ($i = 1, 2, \cdots, n$) and so

$$(x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n})^\alpha = (x_1')^{i_1} (x_2')^{i_2} \cdots (x_n')^{i_n} \qquad (***)$$

where i_1, i_2, $\cdots$, i_n are non-negative integers and $i_1 + i_2 + \cdots + i_n \leq 2$.

With the same argument by which we prove that $(*)$ holds for any non-negative integers $i_1, i_2, \cdots, i_n$, we have $(***)$ holds for any non-negative integers $i_1, i_2, \cdots, i_n$. Thus $\alpha \in \mathrm{Aut}(G)$, so $S \cup S^{-1}$ is CI. Thus (1) of the theorem holds.

Remember that $X = \mathrm{Cay}(G, S)$ and $\overline{X} = \mathrm{Cay}(G, S \cup S^{-1})$. Let $A = \mathrm{Aut}(X)$ and $\overline{A} = \mathrm{Aut}(\overline{X})$. Godsil [8] proved that $N_A(R(G)) = R(G) \cdot \mathrm{Aut}(G, S)$. It is easy to see that X is normal if and only if $A_1 = \mathrm{Aut}(G, S)$, where A_1 is the stabilizer of the identity element 1 in A. Similarly $\overline{X}$ is normal if and only if $\overline{A}_1 = \mathrm{Aut}(G, S \cup S^{-1})$.

Taking $T = S$ and $\overline{T} = S \cup S^{-1}$ respectively in the proofs of S and $S \cup S^{-1}$ being CI, the same arguments will give $\sigma \in \mathrm{Aut}(G)$ and $\alpha \in \mathrm{Aut}(G)$, which means that $A_1 \leq \mathrm{Aut}(G, S)$ and $\overline{A}_1 \leq \mathrm{Aut}(G, S \cup S^{-1})$. The converses are obvious. This finishes the proof of (2). □

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