A Note on Isomorphisms of Cayley Digraphs of Abelian Groups *

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Abstract

Let S and T be two minimal generating subsets of a finite abelian group G. We prove that if $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ then there exists an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$.

Let G be a finite group and S a subset of G not containing the identity element 1. The Cayley digraph $X = \mathrm{Cay}(G,S)$ of G with respect to S is defined by

$$V(X) = G,$$
$$E(X) = \{(g, sg) \mid g \in G, s \in S\}.$$

It is easy to check that if α is an automorphism of G then $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,S^\alpha)$. Conversely, we call a subset S of G a CI-subset, if for any subset T of G with $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$, there is an automorphism α of G such that $S^\alpha = T$. The concept of a CI-subset is important for the study of the isomorphism problem for Cayley digraphs. However, there are not many results in the literature about this concept. The second author conjectured at the third China–US conference on graph theory in 1993 that every minimal generating subset of a finite group is a CI-subset, see also [3, Problem 8]. Huang and Meng [1] verified it for cyclic groups, and Li [2] verified it for abelian groups of odd order. However, in the same paper Li found a counterexample for an abelian group of even order. Namely, Li proved

Proposition *(1) Every minimal generating subset of an abelian group of odd order is CI.*

(2) Let $G = \langle a \rangle \times \langle x \rangle \times \langle e \rangle \cong Z_3 \times Z_4 \times Z_2$ *and let* $S = \{x, xe, ax^2\}$ *and* $T = \{x, xe, ax^2e\}$. *Then* S *is a minimal generating subset of* G *and the Cayley digraph*

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Cay(G,S) *is isomorphic to* Cay(G,T). *However, there is no automorphism of* G *which maps* S *to* T. *In other words, neither* S *nor* T *are CI.*

In Li's paper there is a small mistake. Li claimed that both the sets S and T in above proposition are minimal generating subsets of G. However, it is easy to check that S is, but T is not.

The purpose of this note is to prove the following theorem.

Theorem *Let* G *be a finite abelian group and let both* S *and* T *be minimal generating subsets of* G. *Let* $X = \text{Cay}(G,S)$ *and* $Y = \text{Cay}(G,T)$ *be isomorphic. Then there exists an* $\alpha \in \text{Aut}(G)$ *such that* $S^\alpha = T$.

As a consequence of this, for abelian groups, all generating subsets with the minimum number of generators are CI. That is the above-mentioned conjecture is true for minimum generating subsets of abelian groups.

Proof of Theorem: If $|S| = 1$, then G is cyclic and the theorem holds. So we may assume that $|S| > 1$ in what follows.

Take an isomorphism σ from Cay(G,S) to Cay(G,T) with $1^\sigma = 1$. Then $T = S^\sigma$. Now we define an equivalence relation "$\sim$" on G by

$$g \sim h \iff g^2 = h^2, \ \forall g,h \in G.$$

Then the restriction of "$\sim$" to S (or to T) is also an equivalence relation on S (or on T). Let $S_1, \cdots, S_k$ be the equivalence classes of S under (the restriction of) the relation "$\sim$" on S. Then $S = \bigcup_{i=1}^k S_i$. Let $T_i = S_i^\sigma$, for $i = 1, \cdots, k$. We have

Fact 1: $T_1, \cdots, T_k$ are also the equivalence classes of T under the relation "$\sim$".

To see this, first we have the following.

Observation 2: For any $s_1, s_2 \in S$, $s_1 \neq s_2$, we have the intersection of the out-neighborhoods $X_1(s_1)$ and $X_1(s_2)$ of s_1 and s_2 in the graph X is

$$X_1(s_1) \cap X_1(s_2) = \begin{cases} \{s_1 s_2\}, & \text{when } s_1 \not\sim s_2, \\ \{s_1 s_2, s_1^2\}, & \text{when } s_1 \sim s_2. \end{cases} \tag{1}$$

Proof of Observation 2: Obviously, $X_1(s_1) \cap X_1(s_2)$ contains $\{s_1 s_2\}$ or $\{s_1 s_2, s_1^2\}$ depending on $s_1 \not\sim s_2$ or $s_1 \sim s_2$, respectively. To prove the reverse inclusion, we assume that there is another common out-adjacent vertex of s_1 and s_2, say $x = s_1 s = s_2 s'$ where $s, s' \in S$. Then at least one of s and s', say s, is not in $\{s_1, s_2\}$. It follows that $s = s_1^{-1} s_2 s'$ and G can be generated by $S \setminus \{s\}$, contradicting the minimality of S. □

By symmetry, for any $t_1, t_2 \in T$, $t_1 \neq t_2$, we also have

$$Y_1(t_1) \cap Y_1(t_2) = \begin{cases} \{t_1 t_2\}, & \text{when } t_1 \not\sim t_2, \\ \{t_1 t_2, t_1^2\}, & \text{when } t_1 \sim t_2. \end{cases} \tag{2}$$

Now we are ready to prove Fact 1. Given $s_1, s_2 \in S$, set $t_i = s_i^\sigma$ for $i = 1, 2$. By (1) and (2), $s_1 \sim s_2 \iff |X_1(s_1) \cap X_1(s_2)| = 2 \iff |(X_1(s_1) \cap X_1(s_2))^\sigma| = |Y_1(t_1) \cap Y_1(t_2)| = 2 \iff t_1 \sim t_2$, so Fact 1 holds.

Next, we consider the image of a product $s_1 s_2 \cdots s_n$ under σ, where $s_i \in S$ for $i = 1, 2, \cdots, n$. We shall prove the following.

Fact 3: $(s_1 s_2 \cdots s_n)^\sigma = y_1 y_2 \cdots y_n$ for some $y_i \in T$, $i = 1, 2, \cdots, n$, with $y_i \sim s_i^\sigma$ $\forall i$.

Proof Note that when $n = 1$, the conclusion is trivially true. So, by induction, it suffices to prove that for any $x \in G$ and $s_1, s_2 \in S$, if $(s_i x)^\sigma = y_i x^\sigma$, $i = 1, 2$, where $y_i \sim s_i^\sigma$, then $(s_1 s_2 x)^\sigma = z_1 z_2 x^\sigma$, where $z_i \sim s_i^\sigma$ for $i = 1, 2$. To prove this, we distinguish two cases: (i) $s_1 \not\sim s_2$, and (ii) $s_1 \sim s_2$. In the first case, we also have $y_1 \not\sim y_2$ (by Fact 1). Since, by the same argument as in the proof of Observation 2, $s_1 s_2 x$ is the only common out-adjacent vertex of $s_1 x$ and $s_2 x$, $y_1 y_2 x^\sigma$ is also the only common out-adjacent vertex of $y_1 x^\sigma$ and $y_2 x^\sigma$, so $(s_1 s_2 x)^\sigma = y_1 y_2 x^\sigma$ and the conclusion holds. In the second case, we also have $y_1 \sim y_2$ (by Fact 1). In this case $s_1 x$ and $s_2 x$, (resp. $y_1 x^\sigma$ and $y_2 x^\sigma$), have two common out-adjacent vertices $s_1 s_2 x$ and $s_1^2 x$, (resp. $y_1 y_2 x^\sigma$ and $y_1^2 x^\sigma$), so $(s_1 s_2 x)^\sigma = y_1 y_2 x^\sigma$ or $y_1^2 x^\sigma$. The conclusion also holds. □

Now we are in the position to prove the main theorem.

Let $S_i = \{s_{i1}, s_{i2}, \cdots, s_{in_i}\}$ for $i = 1, 2, \cdots, k$. Let $s_{ij}^\sigma = t_{ij}$ for $i = 1, \cdots, k$, and $j = 1, \cdots, n_i$. Then $T_i = S_i^\sigma = \{t_{i1}, t_{i2}, \cdots, t_{in_i}\}$, and $S = \bigcup_{i=1}^k S_i$, $T = \bigcup_{i=1}^k T_i$. Now we define a mapping $\alpha : G \to G$ by

$$\prod_{i=1}^{k} \prod_{j=1}^{n_i} s_{ij}^{e_{ij}} \mapsto \prod_{i=1}^{k} \prod_{j=1}^{n_i} t_{ij}^{e_{ij}} \quad \text{where } e_{ij} \text{ are arbitrary non-negative integers.}$$

We shall prove that α is an automorphism of G and hence complete the proof of the theorem. Note that S and T are generating subsets of G. It suffices to prove that α is well-defined. To do so, we need to prove

Fact 4: If $\prod_{i=1}^k \prod_{j=1}^{n_i} s_{ij}^{e_{ij}} = 1$, then $\prod_{i=1}^k \prod_{j=1}^{n_i} t_{ij}^{e_{ij}} = 1$.

Proof Assume that

$$\prod_{i=1}^{k} \prod_{j=1}^{n_i} s_{ij}^{e_{ij}} = 1. \tag{3}$$

First we observe that if S_i has at least two elements for some i, then the exponent e_{ij} in above product is even for any $j = 1, \cdots, n_i$. (For if e_{ij} is odd for some j, then $s_{ij}^{e_{ij}} = s_{ij} s_{ij'}^{e_{ij}-1}$ for any $j' \neq j$ since $s_{ij}^2 = s_{ij'}^2$. Replacing $s_{ij}^{e_{ij}}$ by $s_{ij} s_{ij'}^{e_{ij}-1}$ in (3), we can express s_{ij} as a product of elements in $S \setminus \{s_{ij}\}$, contradicting the minimality of S.) Again since $s_{ij}^2 = s_{ij'}^2$ for any i, j, j', we have

$$\prod_{i=1}^{k} \prod_{j=1}^{n_i} s_{ij}^{e_{ij}} = \prod_{i=1}^{k} s_{i1}^{e_i} = 1, \tag{4}$$

where $e_i = \sum_{j=1}^{n_i} e_{ij}$. Similarly, we have

$$\prod_{i=1}^{k}\prod_{j=1}^{n_i} t_{ij}^{e_{ij}} = \prod_{i=1}^{k} t_{i1}^{e_i}. \tag{5}$$

Let σ act on (3), then by Fact 3 we have

$$\left(\prod_{i=1}^{k}\prod_{j=1}^{n_i} s_{ij}^{e_{ij}}\right)^{\sigma} = \prod_{i=1}^{k}\prod_{j=1}^{n_i} t_{ij}^{f_{ij}} = 1, \tag{6}$$

where $\sum_{j=1}^{n_i} f_{ij} = e_i$ and f_{ij} is even if $|T_i| = |S_i| > 1$. So we have

$$\prod_{i=1}^{k} t_{i1}^{e_i} = 1. \tag{7}$$

Combining (5) and (7) we have

$$\prod_{i=1}^{k}\prod_{j=1}^{n_i} t_{ij}^{e_{ij}} = 1,$$

as required. □

Remarks: If G has odd order, every S_i will have a single element. By the same argument as in the proof of Fact 3, we can see that σ itself preserves the multiplication without assuming that T is minimal. So S is CI. Thus we get Li's result, the Proposition (1).

References

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