Isomorphisms of P_4-graphs*

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Abstract

For graphs G and G' with minimum degree $\delta = 3$ and satisfying one of two other conditions, we prove that any isomorphism from the P_4-graph $P_4(G)$ to $P_4(G')$ can be induced by a vertex-isomorphism of G onto G'. We also prove that a connected graph G is isomorphic to its P_4-graph $P_4(G)$ if and only if G is a cycle of length at least 4.

1. Introduction.

Broersma and Hoede [1] generalized the concept of line graphs and introduced the concept of path graphs. We follow their terminology and give the following definition. Denote by $\Pi_k(G)$ the set of all paths of G on k vertices ($k \geq 1$). The *path graph* $P_k(G)$ of a graph G has vertex set $\Pi_k(G)$ and edge set $\mathcal{E}_k(G)$ with the property that for any $H, K \in \Pi_k(G)$ with $H = x_1x_2\cdots x_k$ and $K = y_1y_2\cdots y_k$ there is an edge $HK \in \mathcal{E}_k(G)$ if and only if $x_i = y_{i+1}$ or $y_i = x_{i+1}$ for $1 \leq i \leq k-1$. The way of describing a line graph stresses the adjacency concept, whereas the way of describing a path graph stresses the concept of path generation by consecutive paths.

For a graph transformation, there are two general problems [2]. We state them here for the P_4-transformation.

Characterization Problem: Characterize those graphs that are the P_4-graph of some graph.

* Partially supported by NSFC, the Third World Academy of Sciences and the Institute of Mathematics, National Autonomous University of Mexico .

Determination Problem: Determine which graphs have a given graph as their P_4-graph.

For P_2-graphs, i.e., line graphs, there is a well-known result concerning the Determination Problem: If G and G' are connected and have isomorphic line graphs, then G and G' are isomorphic unless one is $K_{1,3}$ and the other is K_3. This result is due to Whitney [3]. For the Determination Problem of P_3-graphs, Broersma and Hoede found two pairs and two classes of nonisomorphic connected graphs with isomorphic connected P_3-graphs, see [1]. These examples suggest that to obtain a similar counterpart with respect to P_3-graphs for Whitney's result on line graphs seems to be very difficult. In [4] we proved that the P_3-transformation is one-to-one on all graphs with $\delta \geq 4$. Later in [7] we obtained the same result for all graphs with $\delta \geq 3$. Recently, we proved [8] that for $k \geq 4$ the P_k-transformation is one-to-one on all graphs with minimum degree $\delta \geq k$. Moreover, we proved that for such graphs any P_k-isomorphism can be induced by a vertex-isomorphism.

In this paper, we shall focus our attention on P_4-isomorphisms. We shall ask the question whether for $\delta = 3$ every P_4-isomorphism can be induced by a vertex-isomorphism. We find that it turns out to be untrue. At the end of Section 3, we shall show that there is a P_4-isomorphism from $P_4(K_4)$ to itself that cannot be induced by any vertex-isomorphism of K_4 onto itself, where K_4 is the complete graph with 4 vertices. Unfortunately, the P_4-graph of K_4 is $3C_4$, where $3C_4$ is the graph obtained by taking three disjoint copies of C_4 together, which is not connected. However, at the moment we do not know if there are any connected graphs with minimum degree $\delta = 3$ for which the P_4-graphs are connected and we can find a P_4-isomorphism that cannot be induced by any vertex-isomorphism. It would be very interesting to find such graphs. In Section 3, we shall prove that for many graphs with minimum degree $\delta = 3$, every P_4-isomorphism can be induced by a vertex-isomorphism.

Finally, in Section 4, we shall consider the question of which graphs G have $P_4(G) \cong G$. We prove that G must be a cycle of length at least 4, a result similar to that for P_3-transformations (see Theorem 3.1 of [1]). But, our proof is a little bit more complicated. It seems not to be easy to extend the same proof-technique to show that $P_k(G) \cong G$ implies that G is a cycle of length at least k for general $k \geq 5$.

2. Preliminaries.

In what follows, all graphs are connected and simple with at least 5 vertices. As usual, $d(u)$ denotes the degree of a vertex u and $N(u)$ denotes the neighborhood of u. For a nonnegative integer d, we denote by $\mathcal{G}_d$ the class of all connected graphs with minimum degree at least d. An edge is called an *endedge* if it is incident with an endvertex.

We will follow the treatment of [4] for P_3-graphs, which in turn reflects Jung's ideas in [5] and Beineke-Hemminger's treatment in [6]. We introduce the following notation and obtain the corresponding results.

A *vertex-isomorphism* from G to G' is a bijection $f : V(G) \to V(G')$ such that two vertices are adjacent in G if and only if their images are adjacent in G'. We let $\Gamma(G, G')$ denote the set of all vertex-isomorphisms of G to G'.

An *edge-isomorphism* from G to G' is a bijection $f : E(G) \to E(G')$ such that two edges are adjacent in G if and only if their images are adjacent in G'. Obviously, an edge-isomorphism of two graphs is exactly a vertex-isomorphism of their line graphs. We let $\Gamma_e(G, G')$ denote the set of all edge-isomorphisms of G to G'.

We shorten $\Gamma(P_4(G), P_4(G'))$ to $\Gamma_4(G, G')$ and call the members P_4- *isomorphisms* from G to G'.

For $f \in \Gamma_e(G, G')$, define a mapping f^* by $f^*(tuvw) = f(tu)f(uv)f(vw)$ for a P_4-path $tuvw$ in G, and call f^* the *mapping induced by* f. We let $\Gamma^*(G, G') = \{f^* | G \in \Gamma_e(G, G')\}$.

Note that f^* is not defined for a connected graph in general unless it has at least one P_4-path. Also note that the two edge-isomorphisms of the graph P_4 induce the same $*$-function.

Theorem 1. *If $G, G' \in \mathcal{G}_3$, then*
(1) $\Gamma^*(G, G') \subseteq \Gamma_4(G, G')$.
(2) the mapping $T : \Gamma_e(G, G') \to \Gamma^*(G, G')$ given by $T(f) = f^*$ is one to one.

Proof. (1) Let $tuvw$ be a P_4-path in G and $f \in \Gamma_e(G, G')$. Then $f(tu)$, $f(uv)$, $f(vw) \in E(G')$. Since f preserves adjacency and non-adjacency, we have that $f(tu)f(uv)f(vw)$ is a P_4-path in G', i.e., f^* is a mapping from $\Pi_4(G)$ to $\Pi_4(G')$. Obviously, f^* is a bijection. Since f is an edge-isomorphism, we know that $f^* \in \Gamma_4(G, G')$, i.e., $\Gamma^*(G, G') \subseteq \Gamma_4(G, G')$.

To prove (2), let $f_1, f_2 \in \Gamma_e(G, G')$ and $f_1 \neq f_2$. Then there exists an edge uv such that $f_1(uv) \neq f_2(uv)$. Since $G \in \mathcal{G}_3$, we can find a P_4-path $tuvw$ such that ${f_1}^*(tuvw) \neq {f_2}^*(tuvw)$. Thus, the mapping T is one to one. ■

If $P_4 = tuvw$, then the edge uv is called the *middle edge* of the P_4 and $tuvw = wvut$. We let $S(uv)$ denote the set of all P_4-paths with a common middle edge uv. Any subset of $S(uv)$ is called a *double star* at the edge uv. A mapping $f : \Pi_4(G) \to \Pi_4(G')$ is called *double star-preserving* if the set $f(S(uv))$ is a double star in G' for every edge uv of G. Let f be a double star-preserving P_4-isomorphism from G to G'. Then, if two P_4-paths form a P_5-path, their images under f do the same.

Theorem 2. *Let $G, G' \in \mathcal{G}_3$ and let $f : \Pi_4(G) \to \Pi_4(G')$ be a bijective mapping. Then f is induced by an edge-isomorphism from G to G' if and only if f and f^{-1} are double star-preserving P_4-isomorphisms.*

Proof. The condition is clearly necessary. For the sufficiency, suppose that f and f^{-1} are double star-preserving P_4-isomorphisms. Thus, for each edge uv in G, there exists an edge $u'v'$ in G' such that $f(S(uv)) \subseteq S(u'v')$. Moreover, $u'v'$ is uniquely determined by uv. Otherwise, let $f(S(uv)) \subseteq S(u'v')$ and $f(S(uv)) \subseteq S(u''v'')$. If $u'v' \neq u''v''$, then $f(Suv)) \subseteq S(u'v') \cap S(u''v'') = \emptyset$. Since $G \in \mathcal{G}_3$, then $f(S(uv)) \neq \emptyset$. This is a contradiction. Since $f(S(uv)) \subseteq S(u'v')$ and $G' \in \mathcal{G}_3$, we must have $f^{-1}(S(u'v')) \subseteq S(uv)$. Therefore, $f(S(uv)) = S(u'v')$ and

$f^{-1}(S(u'v')) = S(uv)$. We conclude that the function f determines a well-defined function $\widetilde{f} : E(G) \to E(G')$ for which $f(S(uv)) = S(\widetilde{f}(uv))$. It is not difficult to see that $\widetilde{f}$ is a bijection. Now we prove that $\widetilde{f}$ preserves adjacency and nonadjacency. In fact, if tuv is a P_3-path in G, then there is a P_4-path in $S(tu)$ adjacent to some P_4-path in $S(uv)$. Since f is a P_4-isomorphism and $f(S(tu)) = S(\widetilde{f}(tu))$ as well as $f(S(uv)) = S(\widetilde{f}(uv))$, there exists a P_4-path in $S(\widetilde{f}(tu))$ adjacent to some P_4-path in $S(\widetilde{f}(uv))$. This implies that $\widetilde{f}(tu)$ is adjacent to $\widetilde{f}(uv)$ in G'. Since f^{-1} enjoys the same properties as f, $\widetilde{f}$ also preserves nonadjacency. Finally, we prove that f is induced by $\widetilde{f}$. Let $tuvw$ be a P_4-path and let $xtuv \in S(tu)$. Since f is double star-preserving, we have that $f(xtuv) \in f(S(tu)) = S(\widetilde{f}(tu))$ and $f(xtuv)$ is adjacent to $f(tuvw) \in S(\widetilde{f}(uv))$. Thus, $\widetilde{f}(tu)\widetilde{f}(uv)$ is the common P_3-path of $f(xtuv)$ and $f(tuvw)$. By symmetry, $\widetilde{f}(uv)\widetilde{f}(vw)$ is the other P_3-path of $f(tuvw)$ and hence $f(tuvw) = \widetilde{f}(tu)\widetilde{f}(uv)\widetilde{f}(vw)$. The proof is complete. ■

Lemma 3. *Let $G, G' \in \mathcal{G}_3$ and let f be a P_4-isomorphism from G to G'. Assume G and G' satisfy one of the following conditions:*
(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$,
(2) G and G' do not contain any C_4 as asubgraph.
Then f is double star-preserving if and only if for every P_3-path tuv of G, $f(x_1tuv), \cdots, f(x_rtuv)$ have a common middle edge and $f(tuvy_1), \cdots, f(tuvy_s)$ have a common middle edge, where $x_i \in N(t) \setminus \{u, v\}$ for $1 \leq i \leq r$, $y_j \in N(v) \setminus \{u, v\}$ for $1 \leq j \leq s$.

Proof. The condition is obviously necessary. Let uv be any edge of G and let $tuvw$, $t'uvw'$ be two P_4-paths in $S(uv)$. We will distinguish the following four possible cases. See Figure 1.

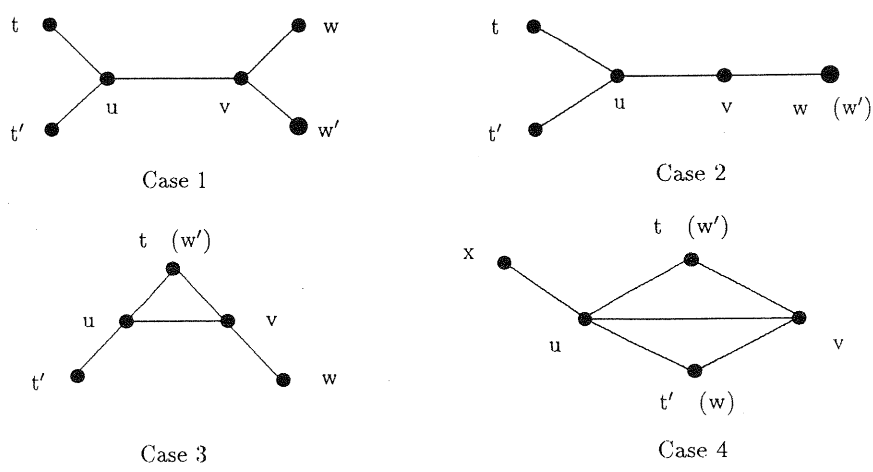


Figure 1

Case 1. The four vertices t, t', w and w' are pairwise different.

From the condition we know that $f(tuvw)$ and $f(tuvw')$ have a common middle edge, and $f(tuvw')$ and $f(t'uvw')$ have a common middle edge. Thus, $f(tuvw)$ and $f(t'uvw')$ have a common middle edge.

Case 2. $t = t'$ or $w = w'$.

By the condition, we know that $f(tuvw)$ and $f(t'uvw')$ have a common middle edge.

Case 3. $t = w'$ but $t' \neq w$, or $t' = w$ but $t \neq w'$.

By a proof similar to that of Case 1, we can show that $f(tuvw)$ and $f(t'uvw')$ have a common middle edge.

Case 4. $t = w'$ and $t' = w$.

If G and G' satisfy condition (1), then there exsits a vertex $x \in N(u) \setminus \{t, v, t'\}$. By the condition, we know that $f(tuvw)$ and $f(xuvw)$ have a common middle edge, $f(xuvw)$ and $f(xuvw')$ have a common middle edge, and $f(xuvw')$ and $f(t'uvw')$ have a common middle edge. Thus, $f(tuvw)$ and $f(t'uvw')$ have a common middle edge. If G and G' satisfy condition (2), then this case cannot occur.

To sum up the above cases, we know that $f(S(uv))$ is a double star of G', i.e., f is double star-preserving. The proof is complete. ■

Note that condition (1) can be weakened as follows: if uv is an edge of a triangle of G, then one of $d(u)$ and $d(v)$ is at least 4.

3. Main Results.

From [8], we have the following two results.

Lemma 4. *Let $f \in \Gamma_4(G, G')$ and let x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ be four P_4-paths of G. Then $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge if and only if $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge.*

Lemma 5. *Let $f \in \Gamma_4(G, G')$ and let x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ be four P_4-paths of G. If $f(x_1tuv)$ and $f(x_2tuv)$ have no common middle edge then $f(x_1tuv)$, $f(x_2tuv)$, $f(tuvy_1)$ and $f(tuvy_2)$ form a C_4 in G'.*

Theorem 6. *Let $G, G' \in \mathcal{G}_3$. Assume G and G' satisfy one of the following two conditions:*
(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$,
(2) G and G' do not contain any C_4 as a subgraph.
Then $f \in \Gamma_4(G, G')$ if and only if f is induced by an edge-isomorphism from G to G', i.e., $P_4(G)$ is isomorphic to $P_4(G')$ if and only if the line graph $L(G)$ is isomorphic to $L(G')$.

Proof. From Theorem 4, we only need prove that both f and f' are double star-preserving. Since G has the same property as G', we only need to show that f is double star-preserving.

The "*if*" part is obvious. In the following we will prove the "*only if*" part. We only need to show that f satisfies the condition of Lemma 3. Let tuv be a P_3-path in

G, $x_1tuv, \cdots, x_mtuv$ and $tuvy_1, \cdots, tuvy_n$ be P_4-paths of G, where $x_i \in N(t) \setminus \{u, v\}$ for $1 \leq i \leq m$, $y_j \in N(v) \setminus \{u, t\}$ for $1 \leq j \leq n$.

If G and G' satisfy condition (1), then $m \geq 2$ and $n \geq 2$. Without loss of generality, we consider $f(x_1tuv)$, $f(x_2tuv)$, $f(tuvy_1)$ and $f(tuvy_2)$. Suppose that $f(x_1tuv)$ and $f(x_2tuv)$ do not have a common middle edge. By Lemma 5, $f(x_1tuv)$, $f(x_2tuv)$, $f(tuvy_1)$ and $f(tuvy_2)$ form a C_4 in G' (denoted by C'=$abcda$), say $f(x_1tuv)$=$abcd$, $f(x_2tuv)$=$cdab$, $f(tuvy_1)$=$bcda$ and $f(tuvy_2)$=$dabc$. Since G and G' satisfy condition (1), there are two vertices $p, q \in N(x_1)$ and a vertex $z \in N(u) \setminus \{v\}$ such that px_1tu, qx_1tu and x_1tuz are P_4-paths in G. If $f(x_1tuv)$ and $f(x_1tuz)$ have a common middle edge, and both $f(x_1tuv)$ and $f(x_1tuz)$ are adjacent to $f(px_1tu)$, we have that $f(x_1tuv)$ and $f(x_1tuz)$ have a common P_3-path, say abc, and $f(x_1tuz)$=$abcd'$. So $f(x_1tuz)$ is adjacent to $f(tuvy_2)$, but x_1tuz is not adjacent to $tuvy_2$ in G, a contradiction to the fact that $f \in \Gamma_4(G, G')$. If $f(x_1tuv)$ and $f(x_1tuz)$ have no common middle edge, by Lemma 5, $f(x_1tuv)$, $f(x_1tuz)$, $f(px_1tu)$ and $f(qx_1tu)$ form a C_4 in G' (denoted by C''). Obviously, C'=C'', so we have $f(x_1tuz)$=$f(x_2tuv)$, a contradiction. Then $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge. From Lemma 4, we have that $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge.

If G and G' satisfy condition (2), we distinguish the following three cases.

Case 1. $m \geq 2$ and $n \geq 2$.

Without loss of generality, we consider $f(x_1tuv)$, $f(x_2tuv)$, $f(tuvy_1)$ and $f(tuvy_2)$. Suppose that $f(x_1tuv)$ and $f(x_2tuv)$ do not have a common middle edge. By Lemma 5, $f(x_1tuv)$, $f(x_2tuv)$, $f(tuvy_1)$ and $f(tuvy_2)$ form a C_4 in G', a contradiction. Then $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge, and $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge.

Case 2. $m = 1$ and $n \geq 2$ (or $n = 1$ and $m \geq 2$).

If $m = 1$, the edge tv must belong to $E(G)$. Since G does not contain any C_4 as a subgraph, there are two vertices $p, q \in N(x_1)$ and a vertex $z \in N(u) \setminus \{t, v\}$ such that px_1tu, qx_1tu and x_1tuz are P_4-paths in G. A proof similar to that of Case 1 shows that $f(x_1tuv)$ and $f(x_1tuz)$ have a common middle edge, and that $f(px_1tu)$ and $f(qx_1tu)$ have a common middle edge. Let $f(x_1tuv)$=$abcd$, then $f(px_1tu)$=$habc$, $f(qx_1tu)$=$kabc$ and $f(x_1tuz)$=$abce$. Since both $f(tuvy_1)$ and $f(tuvy_2)$ are adjacent to $f(x_1tuv)$ but not to $f(x_1tuz)$, then $f(tuvy_1)$=$bcdw$ and $f(tuvy_2)$=$bcdw'$, i.e., $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge.

Case 3. $m = 1$ and $n = 1$.

This case is trival.

To sum up the above cases, we have proved that f is double star-preserving, which completes the proof. ■

From Theorem 3.2 of [6] and our Theorems 1 and 6, the following results are immediate.

Theorem 7. *Let $G, G' \in \mathcal{G}_3$. Assume G and G' satisfy one of the following two conditions:*

(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$,

(2) G and G' do not contain any C_4 as a subgraph.

Then $f \in \Gamma_4(G, G')$ if and only if f is induced by an isomorphism of G to G', i.e., $P_4(G)$ is isomorphic to $P_4(G')$ if and only if G is isomorphic to G'.

Corollary 8. *Let $G, G' \in \mathcal{G}_3$. Assume G and G' satisfy one of the following two conditions:*
(1) if u is a vertex of some triangle in G, then $d(u) \geq 4$,
(2) G and G' do not contain any C_4 as a subgraph.
Then the P_4-transformation is one to one.

Now we show that there is a P_4-isomorphism from $P_4(K_4)$ to itself that cannot be induced by any vertex-isomorphism of K_4 onto itself. The graph K_4 and its P_4-graph $3C_4$ are shown in Figure 2.

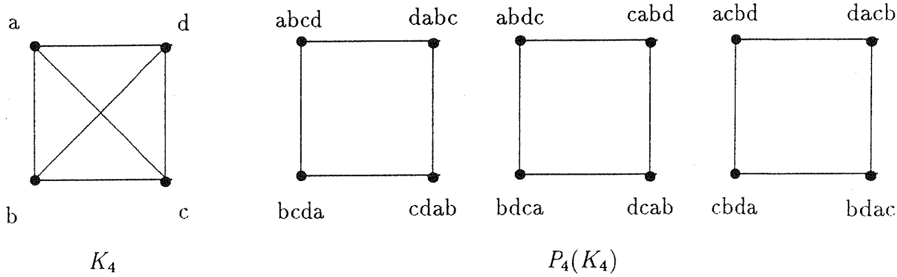


Figure 2

We define a mapping $f : \Pi_4(K_4) \to \Pi_4(K_4)$ by $f(abcd) = cdab$, $f(cdab) = abcd$ and for the other P_4-paths of $\Pi_4(K_4)$, the image of each under f is itself. Obviously, $f \in \Gamma_4(K_4, K_4)$. There are only two automorphisms of K_4, say f_1 and f_2, such that $f_i^*(abcd) = cdab$, $f_i^*(cdab) = abcd$, $i = 1,2$, i.e., $f_1(a) = c$, $f_1(b) = d$, $f_1(c) = a$, $f_1(d) = b$, and $f_2(a) = b$, $f_2(b) = a$, $f_2(c) = d$, $f_2(d) = c$. It is easy to find a P_4-path in $\Pi_4(K_4)$ such that its image under the induced P_4-isomorphism f_i^* ($i = 1, 2$) is not itself. Then the P_4-isomorphism f from $P_4(K_4)$ to itself cannot be induced by any vertex-isomorphism of K_4 onto itself.

4. Fixed Point of a P_4-transformation.

From the definition of P_4-graphs, we have

Lemma 9. *P_4-graphs do not contain triangles.*

Theorem 10. *A connected graph G is isomorphic to its path graph $P_4(G)$ if and only if G is a cycle of length at least four.*

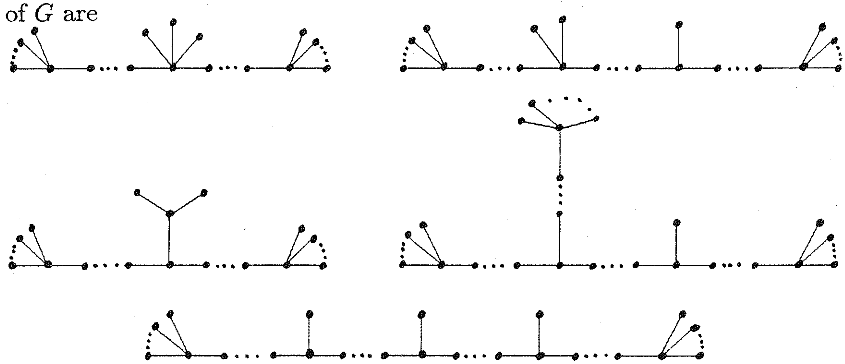
Proof. It is easy to see that the "*if*" part holds.

Let G have n vertices. Then $P_4(G)$ must have n vertices too. So G must have exactly n subgraphs P_4.

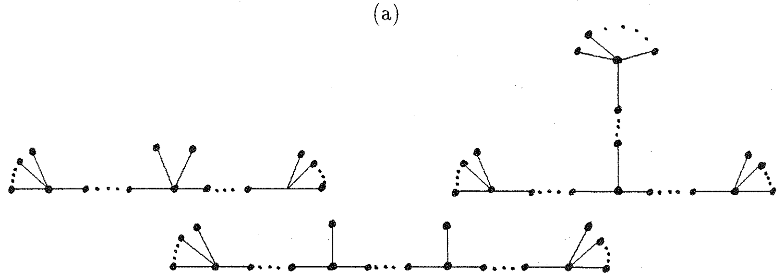
Since G is connected, it has a spanning tree T. Let a longest path in T be $x_1 x_2 \cdots x_{r-1} x_r$ ($r \geq 4$). If $d(x_{r-1}) = m \geq 3$, let $N(x_{r-1}) \setminus \{x_{r-2}, x_r\} =$

$\{x_{r+1}, x_{r+2}, \cdots x_{r+m-2}\}$. If T is transformed into a tree T^* by removing the end-edges $x_{r-1}x_i$ from x_{r-1}, and adding it to the end-vertex x_{i-1}, $i = r+1, \cdots, r+m-2$, then the number of P_4's in T^* is lower than that in T by $(d(x_{r-1})-2)(d(x_{r-2})-2)$, which is non-negative. If $d(x_{r-1}) = 2$, let T_s be a subtree pendant of x_j, $3 \leq j \leq r-2$, and let x be a neighbor of x_j in T_s. If T is transformed into a tree T^* by removing the subtree pendant T_s from x_j and adding it to the end-vertex x_r of the resulting tree, then the number of P_4's in T^* is lower than that in T by $(d(x)-1)(d(x_j)-2)+d(x_{j-1})+d(x_{j+1})-3$, which is positive.

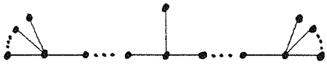
By repetition of the above two transformations, every tree T can be transformed into P_n, which has $n-3$ subgraphs P_4. If T is to have no more than n subgraphs P_4, it cannot therefore have a vertex x_i of degree 6 or more in a longest path $x_1x_2\cdots x_{r-1}x_r$ $(r \geq 4)$, for $3 \leq i \leq r-2$, as the above transformations can make T into a P_n with a change of at least 4 in the number of $P_4's$ and T, and thus G would have at least $(n-1)+4 = n+1$ subgraphs P_4. Similarly, T cannot have two or more vertices x_i of degree 4 or 5, or four or more vertices of degree 3 in its longest path $x_1x_2\cdots x_{r-1}x_r$ $(r \geq 4)$, for $3 \leq i \leq r-2$. And let u be a neighbor of x_i, $3 \leq i \leq r-2$, then $d(u) \leq 3$. If $d(u) = 3$, then there is only one vertex of degree 3 in $\{x_i | 3 \leq i \leq r-2\}$. The remaining possible structures of the spanning tree of G are



(a)



(b)



(c)

(d)

In case (a), the number of subgraphs P_4 is equal to the number of vertices. $P_4(G)$ contains isolated vertices if two adjacent vertices of an edge are incident with two end edges, respectively. By the constitution of P_4-graphs, it can be checked that G cannot be any of these trees.

In cases (b) and (c), an edge has to be added to obtain a graph with at least n paths of length 3. However, by Lemma 9, then at least three subgraphs P_4 are added to the $n-1$ or $n-2$ present in the spanning tree T and $P_4(G)$ would have at least $n+1$ vertices.

In case (d), addition of an edge leads to a unicyclic graph G, since otherwise it belongs to case (b) or (c). When $\alpha \geq 2$ or $\beta \geq 2$, then at least four subgraphs P_4 are added to the $n-3$ present in the spanning tree T and $P_4(G)$ would have at least $n+1$ vertices. When $\alpha = 1$ and $\beta = 1$, if the number of vertices of degree 3 is two, G contains $n+3$ subgraphs P_4, and if this number is one, then G contains $n+1$ subgraphs P_4. The only possibility left is that the added edge is adjacent to two endvertices of T, and G is a cycle of length at least 4. The proof is complete. ■

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(Received 29/4/96)