# On Cross Numbers of Minimal Zero Sequences

Scott T. Chapman*
Trinity University
Department of Mathematics
715 Stadium Drive
San Antonio, Texas 78212-7200

Alfred Geroldinger
Institut für Mathematik
Karl-Franzens-Universität Graz
Heinrichstrasse 36
A-8010 Graz, Austria

**Abstract**

Let $G$ be a finite abelian p-group. We compute the complete set of cross numbers of minimal zero sequences associated with $G$. We also strengthen a result of Krause concerning minimal zero sequences with cross numbers less than or equal to 1.

## 1 Introduction

Let $G$ be an additively written finite abelian group and $S = (g_1, \cdots, g_l)$ a sequence of elements of $G$. For ease of notation, we will also denote $S$ by $S = g_1 \cdots g_l$ and use exponentiation to represent repetition in the sequence. We say that $S$ is a *zero sequence* if

$$\sum_{i=1}^{l} g_i = 0.$$

Further, $S$ is called a *minimal zero sequence* if $\sum_{i \in I} g_i \neq 0$ for each proper subset $\emptyset \neq I \subset \{1, \cdots, l\}$. The *cross number* $k(S)$ of $S$ is defined by

$$k(S) = \sum_{i=1}^{l} \frac{1}{\operatorname{ord}(g_i)}.$$

Let $\mathcal{U}(G)$ represent the set of all minimal zero sequences of $G$ and

$$W(G) = \{k(S) | S \in \mathcal{U}(G)\}$$

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the set of cross numbers of minimal zero sequences. Then

$$K(G) = \exp(G) \max W(G)$$

is called the *cross number of* $G$.

Cross numbers of minimal zero sequences, and in particular the cross number of $G$, have found a great deal of attention in recent literature (see the references). In this note, we determine $W(G)$ for all finite abelian $p$-groups $G$. In section 3 we solve Problem 5 of [1], which deals with special minimal zero sequences $S$ satisfying $k(S) \leq 1$.

Throughout our discussion, we will use notation consistent with that of the papers [4] - [7]: $\mathbf{Z}$ represents the set of integers, $\mathrm{N}_+$ the set of positive integers, and $C_n$ the cyclic group consisting of $n$ elements.

## 2 On W(G)

**Lemma 1** *Let $G = H \oplus C_{2^r}$ be a finite abelian group with $r \geq 1$, $exp(H) = 2^s m$ for some $0 \leq s < r$, $m$ odd, and $exp(G) = 2^r m = n$. For every $B \in \mathcal{U}(G)$ we have that $k(B) = \frac{\lambda}{n}$ for some even $\lambda \in \mathrm{N}_+$.*

**Proof:** Let

$$B = a_1 \cdots a_k b_1 \cdots b_l \in \mathcal{U}(G)$$

be given with $2^r \not| \operatorname{ord}(a_i)$ for $1 \leq i \leq k$ and $2^r | \operatorname{ord}(b_j)$ for $1 \leq j \leq l$. Thus, there are even numbers $n_i$, $1 \leq i \leq k$, such that

$$n = \operatorname{ord}(a_i) n_i$$

and hence,

$$\sum_{i=1}^{k} \frac{1}{\operatorname{ord}(a_i)} = \sum_{i=1}^{k} \frac{n_i}{n} = \frac{\lambda_1}{n}$$

for some even $\lambda_1 \in \mathrm{N}_+$. For every $1 \leq j \leq l$ we set

$$b_j = d_j + c_j$$

with $d_j \in H$ and $c_j \in C_{2^r}$ with $\operatorname{ord}(c_j) = 2^r$. Then,

$$2^{r-1}(c_j + c_{j'}) = 0$$

for all $1 \leq j < j' \leq l$ and thus

$$2^r \not| \operatorname{ord}(b_j + b_{j'}).$$

Since $\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j = 0$, we infer that $l$ is even. There are odd numbers $m_j$, $1 \leq j \leq l$, such that

$$n = \operatorname{ord}(b_j) m_j$$

and hence

$$\sum_{j=1}^{l} \frac{1}{\mathrm{ord}(b_j)} = \sum_{j=1}^{l} \frac{m_j}{n} = \frac{\lambda_2}{n}$$

for some even $\lambda_2 \in \mathbf{N}_+$.◇

Set $W^*(G) = \{k(S) | S \in \mathcal{U}(G) \text{ and } k(S) \leq 1\}$. In the next theorem we determine $W^*(G)$ for all finite abelian groups $G$.

**Theorem 2** *Let $G$ be a finite abelian group of exponent $n = 2^r m$ for some odd $m$ and some $r \geq 0$. If $C_{2^r} \oplus C_{2^r}$ is a subgroup of $G$, then*

$$W^*(G) = \left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq n \right\}.$$

*Otherwise,*

$$W^*(G) = \left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq n, \lambda \text{ even} \right\}.$$

**Proof:** By definition

$$W^*(G) \subseteq \left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq n \right\}.$$

If $C_{2^r} \oplus C_{2^r}$ is not a subgroup of $G$, then Lemma 1 implies that

$$W^*(G) \subseteq \left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq n, \lambda \text{ even} \right\}.$$

In order to prove the reverse inclusion, let $\lambda \in \{2, \ldots, n\}$ be given (if $G$ has no subgroup isomorphic to $C_{2^r} \oplus C_{2^r}$, then suppose that $\lambda$ is even).

**Case 1:** Suppose that $\lambda$ does not divide $n$. Suppose further that $\gcd(\lambda, m) = \alpha$, $\lambda = \alpha\lambda'$, = and $m = \alpha m'$. Then

$$\frac{\lambda}{n} = \frac{\lambda'}{n'}$$

with $n' = 2^r m', 2 \leq \lambda' \leq n', \lambda \equiv \lambda' \bmod 2$ and $\gcd(\lambda', m') = 1$.

If $\lambda'$ is even, then

$$B = \begin{pmatrix} 1 + m'\mathbf{Z} \\ 1 + 2^r\mathbf{Z} \end{pmatrix}^{(\lambda'-2)} \begin{pmatrix} 2 + m'\mathbf{Z} \\ 1 + 2^r\mathbf{Z} \end{pmatrix} \begin{pmatrix} -\lambda' + m'\mathbf{Z} \\ (1 - \lambda') + 2^r\mathbf{Z} \end{pmatrix} \in \mathcal{U}(C_{m'} \oplus C_{2^r})$$

and

$$k(B) = \frac{\lambda' - 2}{n'} + \frac{1}{n'} + \frac{1}{n'} = \frac{\lambda}{n}.$$

If $\lambda'$ is odd, then $C_{2^r} \oplus C_{n'}$ is a subgroup of $G$. Furthermore,

$$B = \begin{pmatrix} 1 + n'\mathbf{Z} \\ 1 + 2^r\mathbf{Z} \end{pmatrix}^{(\lambda'-2)} \begin{pmatrix} 2 + n'\mathbf{Z} \\ 1 + 2^r\mathbf{Z} \end{pmatrix} \begin{pmatrix} -\lambda' + n'\mathbf{Z} \\ (1 - \lambda') + 2^r\mathbf{Z} \end{pmatrix} \in \mathcal{U}(C_{n'} \oplus C_{2^r})$$

and

$$k(B) = \frac{\lambda' - 2}{n'} + \frac{1}{n'} + \frac{1}{n'} = \frac{\lambda}{n}.$$

**Case 2:** Suppose that $\lambda = n$. Take some $0 \neq g \in G$ and set $B = g^{\text{ord}(g)}$. Then $k(B) = 1$.

**Case 3:** Suppose that $\lambda < n$ and $\lambda | n$. Then there is a prime $p \in \mathbf{N}_+$ dividing $\lambda$ such that

$$\frac{\lambda}{n} = \frac{p}{p^s l}$$

with $s \geq 1$, $p \not| l$, $p^s l | n$, and $l \in \mathbf{N}_+$. If $\lambda$ is even, we choose $p = 2$.

**Case 3.1:** Suppose $p = 2$. Take some element $g \in G$ of order $p^s l$ and set $B = (-g)g$. Then

$$k(B) = \frac{2}{p^s l} = \frac{\lambda}{n}.$$

**Case 3.2:** Suppose $p \geq 3$ is odd.

**Case 3.2.1:** Suppose that $l$ is odd. Set

$$B = \begin{pmatrix} 1 + p^s\mathbf{Z} \\ 1 + l\mathbf{Z} \end{pmatrix}^{\frac{p-3}{2}} \begin{pmatrix} 1 + p^s\mathbf{Z} \\ -1 + l\mathbf{Z} \end{pmatrix}^{\frac{p-3}{2}} \begin{pmatrix} 1 + p^s\mathbf{Z} \\ -1 + l\mathbf{Z} \end{pmatrix}^{2} \begin{pmatrix} (1-p) + p^s\mathbf{Z} \\ 2 + l\mathbf{Z} \end{pmatrix}.$$

Then $B \in \mathcal{U}(C_{p^s} \oplus C_l)$ and

$$k(B) = \frac{(p-3)/2}{p^s l} + \frac{(p-3)/2}{p^s l} + \frac{2}{p^s l} + \frac{1}{p^s l} = \frac{\lambda}{n}.$$

**Case 3.2.2:** Suppose that $2^t | l$ for some $t \geq 1$. Since $p$ is odd, $\lambda$ is odd and hence $C_{p^s l} \oplus C_{2^t} \cong C_{p^s} \oplus C_l \oplus C_{2^t}$ is a subgroup of $G$. Set

$$B = \begin{pmatrix} 1 + p^s\mathbf{Z} \\ 1 + l\mathbf{Z} \\ 0 + 2^t\mathbf{Z} \end{pmatrix}^{\frac{p-3}{2}} \begin{pmatrix} 1 + p^s\mathbf{Z} \\ -1 + l\mathbf{Z} \\ 0 + 2^t\mathbf{Z} \end{pmatrix}^{\frac{p-3}{2}} \begin{pmatrix} 1 + p^s\mathbf{Z} \\ -1 + l\mathbf{Z} \\ 0 + 2^t\mathbf{Z} \end{pmatrix} \begin{pmatrix} 1 + p^s\mathbf{Z} \\ -1 + l\mathbf{Z} \\ -1 + 2^t\mathbf{Z} \end{pmatrix} \begin{pmatrix} (1-p) + p^s\mathbf{Z} \\ 2 + l\mathbf{Z} \\ 1 + 2^t\mathbf{Z} \end{pmatrix}.$$

Then $B \in \mathcal{U}(C_{p^s} \oplus C_l \oplus C_{2^t})$ and

$$k(B) = \frac{p}{p^s l} = \frac{\lambda}{n}. \diamond$$

**Example 3** Let G be any finite abelian group of odd order with exponent $n$. Then Theorem 2 yields (with $r = 0$) that

$$W^*(G) = \left\{\frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}\right\}.$$

In particular, if $G \cong C_{p^s}$ (with $p$ an odd prime), then Krause's result [8] implies that

$$W(G) = \left\{\frac{2}{p^s}, \frac{3}{p^s}, \ldots, \frac{p^s}{p^s}\right\}.$$

In particular, if $G \cong C_{p^s}$ (with $p$ an odd prime), then Krause's result [8] implies that

$$W(G) = \left\{ \frac{2}{p^s}, \frac{3}{p^s}, \ldots, \frac{p^s}{p^s} \right\}.$$

For $p = 2$ the same group as above yields

$$W(G) = \left\{ \frac{2}{2^s}, \frac{4}{2^s}, \ldots, \frac{2^s}{2^s} \right\}.$$

**Theorem 4** *Let $G$ be a finite abelian $p$-group for some prime $p$. Suppose that $G \cong \oplus_{i=1}^r C_{n_i}$ with $1 = n_0 < n_1 \leq \cdots \leq n_r = n$. If $p$ is odd, or if $p = 2$ and $n_{r-1} = n$, then*

$$W(G) = \left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq nr - \sum_{i=1}^{r-1} \frac{n}{n_i} \right\}.$$

*Otherwise,*

$$W(G) = \left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq nr - \sum_{i=1}^{r-1} \frac{n}{n_i}, \lambda \text{ even} \right\}.$$

**Proof:** In either of the two cases above, $W(G)$ is contained in the set on the right side of the equality by [5] and Lemma 1. In order to show the reverse inclusion, we proceed by induction on $r$. For $r = 1$ the assertion follows from Theorem 2. Let $e_1, \ldots, e_r$ be a generating system for $G$ with $\text{ord}(e_i) = n_i$ for $1 \leq i \leq r$.

**Case 1:** $p$ odd (or $p = 2$ and $n_{r-1} = n$). First suppose that $G = C_{2^s} \oplus C_{2^s}$ for some $s \in \mathbb{N}_+$ (i.e., $p = r = 2, n_1 = n = 2^s$). By Theorem 2 we have that

$$\{\frac{\lambda}{n} | 2 \leq \lambda \leq n\} \subseteq W(G) \subseteq \{\frac{\lambda}{n} | 2 \leq \lambda \leq 2n - 1\}.$$

For each $0 \leq l \leq n - 1$, set

$$B_l = e_1^{n-1-l} e_2^{n-1-l} (e_1 + e_2)^{l+1} \in \mathcal{U}(G).$$

We have $k(B_l) = \frac{2n-1-l}{n}$ and hence

$$\{\frac{\lambda}{n} | n \leq \lambda \leq 2n - 1\} \subseteq W(G),$$

which completes the proof for $G = C_{2^s} \oplus C_{2^s}$.

Now suppose that $r \geq 2$ if $p$ is odd (resp. $r \geq 3$ if $p = 2$) and that the result holds for $r - 1$. By the induction hypothesis we have that

$$\left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq n(r-1) - \sum_{i=2}^{r-1} \frac{n}{n_i} \right\} = W(\oplus_{i=2}^r C_{n_i}) \subseteq$$

$$W(\oplus_{i=1}^r C_{n_i}) \subseteq \left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq nr - \sum_{i=1}^{r-1} \frac{n}{n_i} \right\}.$$

Hence, it remains to verify that for every $1 \leq l \leq n-1$ there exists some $B_l \in \mathcal{U}(G)$ with

$$k(B_l) = \sum_{i=1}^{r-1} \frac{n_i - 1}{n_i} + \frac{l+1}{n}.$$

Let $l \in \{1, \ldots, n-1\}$ be given. If $p \not| l$ or $p = 2$, then

$$B_l = \prod_{i=1}^{r-1} e_i^{n_i - 1} e_r^l (e_1 + \cdots + e_{r-1} - l e_r)$$

satisfies the desired condition. If $p | l$ and $p$ odd, then

$$B_l = \prod_{i=1}^{r-1} e_i^{n_i - 1} e_r^{l-1} 2e_r (e_1 + \cdots + e_{r-1} - (l+1) e_r)$$

yields the required value for $k(B_l)$.

**Case 2:** $p = 2$ and $n_{r-1} < n$. Suppose $r \geq 2$. By the induction hypothesis we infer that

$$\left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq n(r-1) - \sum_{i=2}^{r-1} \frac{n}{n_i}, \lambda \text{ even} \right\} = W(\oplus_{i=2}^{r} C_{n_i}) \subseteq$$

$$W(\oplus_{i=1}^{r} C_{n_i}) \subseteq \left\{ \frac{\lambda}{n} | 2 \leq \lambda \leq nr - \sum_{i=1}^{r-1} \frac{n}{n_i}, \lambda \text{ even} \right\}.$$

For $0 \leq l \leq \frac{n}{2} - 1$, set

$$B_l = \prod_{i=1}^{r-1} e_i^{n_i - 1} e_r^{n-1-2l} (e_1 + \cdots + e_{r-1} + (2l+1) e_r) \in \mathcal{U}(G).$$

Then

$$k(B_l) = \sum_{i=1}^{r-1} \frac{n_i - 1}{n_i} + \frac{n - 2l}{n}$$

and the proof is complete. ◇

**Remark 5** Let $G$ be a p-group of odd order. Then, by Theorem 4, every possible value between $\min W(G)$ and $\max W(G)$ may be realized as a cross number of some $S \in \mathcal{U}(G)$. We have been unable to find a non-p-group of odd order that satisfies this property.

# 3 On Zero Sequences $S$ with $k(S) \leq 1$

The following result strengthens Lemma 2 in [8] and solves Problem 5 in [1].

**Theorem 6** *Let $G$ be a finite abelian group and $g$ some nonzero element of $G$. The following conditions are equivalent:*

1. *G is cyclic of prime power order.*

2. *$k(S) \leq 1$ for all $S \in \mathcal{U}(G)$ containing $g$.*

**Proof:**(1 $\Rightarrow$ 2) follows from Theorem 4.
(2 $\Rightarrow$ 1) Assume to the contrary that $G$ is not cyclic of prime power order. Then $G$ is the direct sum of two non-trivial subgroups, say $G = G_1 \oplus G_2$. Hence $g = g_1 + g_2$ with $g_1 \in G_1$, $g_2 \in G_2$ and not both $g_1$ and $g_2$ are equal to zero. Without loss of generality, suppose that $g_1 \neq 0$ and $\mathrm{ord}(g_1) = m > 1$. We consider two cases:
**Case 1:** Suppose that $g_2 \neq 0$. Set $\mathrm{ord}(g_2) = n > 1$ and

$$S = g_1^{(m-1)} g_2^{(n-1)} g.$$

Then $S \in \mathcal{U}(G)$ and

$$k(S) = \frac{m-1}{m} + \frac{n-1}{n} + \frac{1}{\mathrm{lcm}(m,n)} > 1,$$

a contradiction.
**Case 2:** Suppose that $g_2 = 0$. We choose an element $h \in G_2$ with $\mathrm{ord}(h) = n > 1$ and set

$$S = g^{(m-1)} h^{(n-1)} (g+h).$$

As above, we have $S \in \mathcal{U}(G)$ and $k(S) > 1$, a contradiction. $\diamond$

The following corollary offers an alternate proof of the main result of [3].

**Corollary 7** *Let $G$ be a finite abelian group and $g$ some nonzero element of $G$. The following conditions are equivalent:*

1. *either $G = C_2$ and $g = 1 + 2\mathbb{Z}$ or $G = C_4$ and $g = 2 + 4\mathbb{Z}$,*

2. *$k(S) = 1$ for all $S \in \mathcal{U}(G)$ containing $g$.*

**Proof:** Since the proof of (1 $\Rightarrow$ 2) is obvious, we prove only (2 $\Rightarrow$ 1). The previous Theorem implies that $G \cong C_{p^n}$ for some prime $p$ and some $n \in \mathbb{N}_+$. Since

$$k((-g)g) = \frac{2}{\mathrm{ord}(g)} = 1,$$

it follows that $\mathrm{ord}(g) = 2$ and hence $p = 2$. If $n = 1$, then we are done. If $n \geq 2$, we choose an element $h \in G$ with $\mathrm{ord}(h) = 2^n$. Then

$$k(gh(-g-h)) = \frac{1}{2} + \frac{2}{2^n} = 1,$$

implies that $n = 2$.$\diamond$

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