A Degree Condition for the Existence of Connected Factors*

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Abstract

Let G be a connected graph of order n, let f and g be two positive integer functions defined on $V(G)$ satisfying $2 \leq f(v) \leq g(v)$ for each vertex $v \in V(G)$. In this paper we prove that if G has a $[f, g]$-factor F and, moreover, among any three independent vertices of G there are (at least) two vertices with degree sum at least $n - \mu$, then G has a matching M such that M and F are edge-disjoint and $M + F$ is a connected $[f, g+1]$-factor of G, where $\mu = \min\{f(v) : v \in V(G)\}$.

As immediate consequences, the result gives a solution to a problem of Kano on the existence of connected $[a, b]$-factors, and it generalizes theorems of M. Cai.

All graphs under consideration are finite and simple. Let xy denote the edge e joining vertices x and y. We write $H \subseteq G$, if a graph H is a subgraph of G. Given disjoint subsets X and Y of $V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and

$$\begin{aligned}
&\overline{X} = V(G) - X, \\
&E_G(X, Y) = \{xy \in E(G) \mid x \in X,\, y \in Y\}, \\
&e_G(X, Y) = |E_G(X, Y)|\,, \\
&\Delta_G(X) = E_G(X, \overline{X}).
\end{aligned}$$

Given $x \in V(G)$, the set of vertices adjacent to x is said to be the neighborhood of x, denoted by $N_G(x)$, and $d_G(x) = |N_G(x)|$ the degree of x. Let f and g be integer-valued functions defined on $V(G)$ such that $0 \leq f(v) \leq g(v) \leq d_G(v)$ for all $v \in V$. An $[f, g]$-factor of G is defined as a subgraph F of G such that

$$f(v) \leq d_F(v) \leq g(v) \qquad \forall v \in V(G),$$

and an $[f, f]$-factor is abbreviated to an f-factor.

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Let M and F be two edge-disjoint subgraphs of graph G, and let $E_{M\cup F} = E(M) \cup E(F)$, where $E(M)$ and $E(F)$ are the edge sets of M and F respectively. We denote by $M + F$ the subgraph induced by $E_{M\cup F}$.

Other notation and terminology are the same as in [1].

The following results about k-factors or connected $[f, g]$-factors are already known.

Theorem 1 [9] *Let $k \geq 3$ be a positive integer and let G be a connected graph of order n and minimum degree at least k where kn is even and $n \geq 4k - 3$. If for each pair of nonadjacent vertices u and v of G*

$$\max\{d_G(u), d_G(v)\} \geq \frac{n}{2},$$

then G has a k-factor.

Theorem 2 [6] *Let k be a positive integer and let G be a graph of order order n and minimum degree at least k where kn is even and $n \geq 4k - 5$. If for each pair of nonadjacent vertices u, v of G*

$$d_G(u) + d_G(v) \geq n$$

then G has both a Hamiltonian cycle C and a k-factor F. Hence G has a connected $[k, k+2]$-factor $C + F$.

Kano posed the following problems

Problem 3 [7] *Find sufficient conditions for a graph G to have a connected $[k, k+1]$-factor.*

Problem 4 [7] *Find sufficient conditions for a graph G to have a connected $[a, b]$-factor.*

M. Cai has proved the following two results.

Theorem 5 [2] *Let k be an integer ≥ 2 and G be a connected graph of order n. If G has a k-factor F and, moreover, among any three independent vertices of G there are (at least) two with degree sum at least $n - k$, then G has a matching M such that M and F are edge-disjoint and $M + F$ is a connected $[k, k+1]$-factor of G.*

An almost k^--factor in a graph is a factor such that every vertex has degree k except at most one with degree $k - 1$.

Theorem 6 [3] *Let k be an odd integer ≥ 3 and let G be a connected graph of odd order $n \geq 2k + 1$. If G has an almost k^--factor F such that the vertex v^* with $d_F(v^*) = k - 1$ has degree at least $n/2$, and, moreover, among any three independent vertices of G there are (at least) two with degree sum at least $n - k$, then G has a matching M such that M and F are edge-disjoint and $M + F$ is a connected $[k, k+1]$-factor of G.*

The purpose of this paper is to prove the following main result, which extends Theorems 5, 6 and gives a solution to problem 3.

Theorem 7 *Let G be a connected graph of order n, let f and g be two positive integer functions defined on $V(G)$ which satisfy $2 \leq f(v) \leq g(v)$ for each vertex $v \in V(G)$. Let G have an $[f,g]$-factor F and put $\mu = \min\{f(v) : v \in V(G)\}$. Suppose that among any three independent vertices of G there are (at least) two vertices with degree sum at least $n - \mu$. Then G has a matching M such that M and F are edge-disjoint and $M + F$ is a connected $[f, g+1]$-factor of G.*

Proof If F is connected, then F itself is a connected $[f,g]$-factor. This implies that it is also a connected $[f, g+1]$-factor, and theorem trivially holds. So we may assume that F is disconnected.

Let $C_1, C_2, \ldots, C_t$ be the components of F, $t \geq 2$. It is obvious that $|V(C_i)| \geq \mu + 1$. First let

$$M_0 = \emptyset \quad \text{and} \quad W_0 = \{C_1\}.$$

Suppose we have found a matching M_s with s edges and $W_s = G[C_1, C_2, \ldots, C_{s+1}]$ such that $W_s + M_s$ is connected. It is clear that

- each edge of M_s connects two components of W_s and
- M_s would form a subtree with s edges if each component $C_i \in W_s$ were contracted into a single vertex $v(C_i)$.

Denote

$$P = W_s + M_s, \quad U = V(M_s), \quad L = V(P) \setminus U.$$

Assume $s < t - 1$, otherwise M_s is a matching as required. As G is connected, there exists an edge $e = az \in \Delta_G(V(P))$ which connects P with another component C_{s+2} (renumbering if necessary). Let $a \in V(P)$ and $z \in V(C_{s+2})$.

Now suppose that G has no matching M_{s+1} with $s+1$ edges such that a connected $(W_s \cup C_{s+2}) + M_{s+1}$ exists. We shall show that there exist three independent vertices such that the degree sum of each pair of them is less than $n - \mu$, which contradicts the assumption of the theorem.

It is easily seen that

$$\begin{array}{ll} s \geq 1, & \\ f(v) + 1 \leq d_P(v) \leq g(v) + 1 & \text{if } v \in U, \\ f(v) \leq d_P(v) \leq g(v)) & \text{if } v \in L, \\ |L| \geq \frac{\mu - 1}{2}|U| + \mu + 1. & \end{array}$$

The last one comes from $|U| = 2s$ and $|L| + |U| = |V(P)| \geq (s+1)(\mu+1)$. Moreover, we have

$$E_G(v,\, L) = \emptyset \quad \forall\, v \in V(C_{s+2}), \tag{1}$$

otherwise, assuming $e^* \in E_G(v,\, L)$, $M_{s+1} = M_s \cup \{e^*\}$ is a matching as required, a contradiction.

As $az \in E(G)$ and $a \in U$, let $ab \in M_s$ be the edge adjacent to az, let A and B be the components of $P - ab$ such that A contains a and B contains b. By the structure of P, we can choose two components C_i, C_j from W_s satisfying

- $C_i \subseteq A$, $\quad C_j \subseteq B$, and
- $|V(C_i) \cap U| = |V(C_j) \cap U| = 1$.

Choose $x \in V(C_i) \setminus U$ and $y \in V(C_j) \setminus U$. Then

$$E_G(\{x, y\}, \overline{V(P)}) = \emptyset, \tag{2}$$

and

$$E_G(x, V(B) \setminus U) = E_G(y, V(A) \setminus U) = \emptyset. \tag{3}$$

Otherwise let $e \in E_G(x, V(B) \setminus U)$, then $M_{s+1} = (M_s \setminus ab) \cup \{za, e\}$ becomes a matching as required, since $\{ab, za, e\} \cap F = \emptyset$, a contradiction. Similarly, $E_G(y, V(A) \setminus U) = \emptyset$. Then x, y and z are three independent vertices.

Now we shall show that the degree sum of each pair of $\{x, y, z\}$ is less $n - \mu$. Let $M_z = \{uv \in M_s \mid \{u, v\} \cap N_G(z) \neq \emptyset\}$. It is clear that $ab \in M_z$. Let X_z and Y_z respectively denote components of $P - M_z$ containing x and y. Obviously, $X_z \subseteq A$, $Y_z \subseteq B$.

Assertion 1 $\quad e_G(\{z, x\}, \{u, v\}) \leq 3 \quad \forall\, uv \in M_s$.

Suppose that for some $uv \in M_s$

$$\{xu,\ xv,\ zu,\ zv\} \subset E(G).$$

Let u and x be in different components of $P - uv$, then it is easily seen that $M_{s+1} = (M_s \setminus uv) \cup \{xu, zv\}$ would be a matching as required, a contradiction.

Assertion 2 $\quad N_G(x) \subset V(X_z) \cup U$.

By (2) and (3), we need only show $E_G(x, L \cap V(A) \setminus V(X_z)) = \emptyset$ since $x \in V(X_z)$ but $x \notin N_G(x)$. We assume $rx \in E_G(x, L \cap V(A) \setminus V(X_z))$. Then by the definition of X_z there exists an edge $uv \in M_z$ on the paths joining r and x in A. By the definition of M_z, either zu or $zv \in E(G)$, say $zu \in E(G)$. Thus $M_{s+1} = (M_s \setminus uv) \cup \{rx, zu\}$ is a matching as required, a contradiction.

Let $h = |M_s \cap E(X_z)|$. Then $|M_s \setminus E(X_z)| = s - h$, X_z and $P \setminus V(X_z)$ respectively contain $h + 1$ and $s - h$ components of W_s. Because each $|C_\ell| \geq \mu + 1$,

$$|M_s \setminus E(X_z)| = s - h \leq \frac{|V(P) \setminus V(X_z)|}{\mu + 1}.$$

By the definition of X_z, $\; V(X_z) \cap V(M_z) \neq \emptyset$. Then

$$|N_G(x) \cap V(X_z) \setminus V(M_z)| \leq |V(X_z)| - 2.$$

Hence from (1), Assertions 1, 2 and the last two inequalities we get that

$$\begin{aligned} d_G(x)+d_G(z) &= |N_G(x)\cap V(X_z)\setminus V(M_z)| \\ &\quad +|N_G(z)\cap \overline{V(P)}|+e(\{x,z\},\ V(M_s\setminus E(X_z))) \\ &\le |V(X_z)|-2+|\overline{V(P)}|-1+3\,|M_s\setminus E(X_z)| \\ &\le |V(X_z)|+|\overline{V(P)}|-3+\tfrac{3\,|V(P)\setminus V(X_z)|}{\mu+1} \\ &\le |V(G)|-3-\tfrac{\mu-2}{\mu+1}\,|V(B)| \\ &\le n-\mu-1. \end{aligned} \tag{4}$$

Similarly,

$$d_G(y)+d_G(z)\le n-\mu-1. \tag{5}$$

Now let us show

$$d_G(x)+d_G(y)\le n-\mu-1. \tag{6}$$

An argument similar to that used in the proof of Assertion 1 shows that

Assertion 3 $e_G(\{x,y\},\ \{u,v\})\le 3\quad \forall\, uv\in M_s$.

Let $M_{xy}=\{uv\in M_s\,|\,e_G(\{x,y\},\ \{u,v\})=3\}$ and let X_c (resp. Y_c) denote the component of $P-M_{xy}$ containing x (resp. y), and

$$M_c=\Delta_P(V(X_c))\cup\Delta_P(V(Y_c)).$$

Then X_c and Y_c are either identical or disjoint. Clearly $M_c\subseteq M_{xy}$ and

Assertion 4 $e_G(\{x,y\},\ \{u,v\})\le 2\quad \forall\, uv\in M_s\cap(E(X_c)\cup E(Y_c))$.

Now we are going to show

Assertion 5 $E_G(r,\ \{x,y\})=\emptyset\quad \forall\, r\in L\setminus(V(X_c)\cup V(Y_c))$.

Assuming $rx\in E(G)$, then by (3) $r\in V(A)$, $ry\notin E(G)$. Because $r\in L\setminus(V(X_c)\cup(V(Y_c))$ and P is connected, there exists (at least) one edge $uv\in M_c$ on the paths which connects r and y in P. By the definition of M_c, either uy or $vy\in E(G)$, say $uy\in E(G)$. Then $M_{s+1}=(M_s\setminus\{ab,\,uv\})\cup\{az,\,rx,\,uy\}$ would be a matching as required, a contradiction. Similarly, we get $ry\notin E(G)$. Thus the assertion holds.

Let $p=|M_s\cap E(X_c)|$, $q=|M_s\cap E(Y_c)|$. Thus X_c and Y_c contain respectively $p+1$ and $q+1$ components of W_s. Now we shall complete the proof by examining two different cases according to whether X_c and Y_c are identical or disjoint.

Case 1. X_c and Y_c are disjoint.

Then $|M_s\setminus(E(X_c)\cup E(Y_c))|=s-(p+q)$, and $P-(V(X_c)\cup V(Y_c))$ contains $s-(p+q)-1$ components of W_s.

An argument similar to the proof of (4) shows that

$$\begin{aligned}
|M_s \setminus (E(X_c) \cup E(Y_c))| &= s - (p+q) \\
&\leq 1 + \frac{|V(P) \setminus (V(X_c) \cup V(Y_c))|}{\mu + 1}. \\
e_G(\{x,y\},\ L) &\leq |L \cap (V(X_c) \cup V(Y_c))| - 2, \\
e_G(\{x,y\},\ U) &\leq 2\,|M_s \cap (E(X_c) \cup E(Y_c))| + 3\,|M_s \setminus (E(X_c) \cup E(Y_c))|.
\end{aligned}$$

Because $\emptyset \neq \Delta_P(V(X_c)) \subseteq M_c$,

$$|L \cap V(X_c)| + 2\,|M_s \cap E(X_c)| \leq |V(X_c)| - 1.$$

Similarly,

$$|L \cap V(Y_c)| + 2\,|M_s \cap E(Y_c)| \leq |V(Y_c)| - 1.$$

We have

$$\begin{aligned}
d_G(x) + d_G(y) &\leq |L \cap (V(X_c) \cup V(Y_c))| - 2 + 2\,|M_s \cap (E(X_c) \cup E(Y_c))| \\
&\quad + 3\,|M_s \setminus (E(X_c) \cup E(Y_c))| \\
&\leq |V(X_c)| + |V(Y_c)| - 4 \\
&\quad + 3\,(1 + \frac{|V(P) \setminus (V(X_c) \cup V(Y_c))|}{\mu + 1}) \\
&\leq |V(P)| - 1 - \frac{\mu - 2}{\mu + 1}\,|V(P) \setminus (V(X_c) \cup V(Y_c))| \\
&\leq |V(P)| - 1 \\
&\leq n - \mu - 2.
\end{aligned}$$

Therefore in this case the required inequality holds.

Case 2. X_c and Y_c are identical.

So $|M_s \setminus E(X_c)| = s - p$, and $P - V(X_c)$ contains $s - p$ components of W_s. Similarly, we have

$$\begin{aligned}
d_G(x) + d_G(y) &\leq |L \cap V(X_c)| - 1 + 2\,|M_s \cap E(X_c)| + 3\,|M_s \setminus E(X_c)| \\
&\leq |V(X_c)| - 2 + \frac{3\,|V(P) \setminus V(X_c)|}{\mu + 1} \\
&\leq |V(P)| - 2 - \frac{\mu - 2}{\mu + 1}\,|V(P) \setminus V(X_c)| \\
&\leq |V(P)| - 2 \\
&\leq n - \mu - 3.
\end{aligned}$$

The proof of Theorem 7 is completed.

Remark 1. Theorems 5, 6 and some similar results about the existence of connected $[a, b+1]$-factors and connected $[f, f+1]$-factors are natural consequences of Theorem 7. So the result of Theorem 7 is the most general result in this sense.

Remark 2. It was pointed out in [2] and [3] that the condition that the degree sum is at least $n-k$ could not be weakened any further. To see this, let $n \geq 3k+3$ and $G := K_1 \vee (K_k \cup K_{k+1} \cup K_{n-2k-2})$, where $\vee$ and $\cup$ denote join and disjoint union. So the result of Theorem 7 is sharp in this sense.

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