Optimizing Slightly Triangulated Graphs

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Abstract

A graph is called slightly triangulated if it contains no chordless cycle with five or more vertices and every induced subgraph has a vertex whose neighbourhood contains no induced path on four vertices. These graphs generalize triangulated graphs and appear naturally in the study of the intersection graphs of the maximal rectangles of orthogonal polygons. Slightly triangulated graphs are perfect (in the sense of Berge). In this paper we present algorithms for recognizing slightly triangulated graphs, for finding a maximum clique, and for finding an optimal colouring.

1 Introduction

Let P_k denote the chordless path with k vertices, C_k the chordless cycle with k vertices and $\overline{G}$ the complement of the graph G. We write $H \subset G$ if H is an induced subgraph of G. The maximum size of a clique in G is denoted by $\omega(G)$, and the number of vertices in G is denoted by n. The neighbourhood of a vertex x in a subgraph H is denoted by $\Gamma_H(x)$.

In the early sixties Berge [1] defined a graph G to be perfect if for every induced subgraph H of G the chromatic number of H is equal to the largest size of a clique in H. Graphs which played an important role in the development of perfect graph theory are the triangulated graphs. A graph is triangulated if it contains no chordless cycle with four or more vertices. Another characterization is that every induced subgraph of a triangulated graph contains a vertex whose neighbourhood is a clique. Triangulated graphs constitute a large class of perfect graphs with numerous applications. They have been thoroughly studied and efficient algorithms are known for these graphs. The reader is referred to [5] for an introduction to this topic. We introduced in [9] (see also [10]) a generalization of triangulated graphs; we call a graph *slightly triangulated* if it satisfies the following two conditions.

1. $\forall k \geq 5, C_k \not\subset G$

2. $\forall H \subset G, \exists x \in H, P_4 \not\subset \Gamma_H(x)$

Each of these conditions is satisfied by triangulated graphs. Therefore the class of slightly triangulated graphs contains the class of triangulated graphs.

Slightly triangulated graphs are useful in characterizing the intersection graphs of the maximal rectangles of a polyomino (see [8]). Because they can contain $\overline{C}_6$, slightly triangulated graphs do not belong to any of the classical classes of perfect graphs, except maybe the quasi-parity graphs (this is an open problem, see [2] for a comprehensive presentation).

In section 2, we will present an algorithm to recognize slightly triangulated graphs in polynomial time. In section 3, we will present an algorithm (also polynomial) to find a maximum clique in a slightly triangulated graph. In section 4, we will present an algorithm to colour these graphs, and discuss the complexity of this algorithm (which is not polynomial).

2 Recognizing slightly triangulated graphs

To simplify the notation, a vertex with a P_4-free neighbourhood will be called a P_4-free vertex. Deciding whether a graph G is slightly triangulated can be decomposed into two independent sub-problems:

- **Problem 1** : Does G contain an induced cycle with 5 or more vertices ?
- **Problem 2** : Does every induced subgraph contain a P_4-free vertex ?

2.1 A solution to problem 1

The first problem has already been solved by Ryan Hayward in his PhD Thesis. He designed an algorithm to test whether a graph contains an induced cycle whose length is at least five. The idea is to test for every induced P_3 abc if the endpoints a and c of the P_3 belong to the same connected component of the graph obtained by removing $\Gamma(b) \setminus \{a, c\}$ and $\Gamma(a) \cap \Gamma(c)$ (including b). If a and c belong to the same connected component, then the P_3 we were considering belongs to a chordless cycle of length at least five. To test whether two vertices belong to the same component can be done in $\mathcal{O}(n^2)$ time (this problem is linear with the number of edges). To list all the P_3's can be done in $\mathcal{O}(n^3)$ time. Therefore the overall complexity is in $\mathcal{O}(n^5)$. A slightly faster algorithm, designed by Jeremy Spinrad [12], runs in $\mathcal{O}(n^{4.376})$.

2.2 A solution to problem 2

The second problem is equivalent to

- **Problem 2'**: Is there a numbering $v_1, \ldots, v_n$ of the vertices of G such that if G_i denotes the subgraph generated by $v_1, v_2, \ldots, v_i$ then $P_4 \not\subset \Gamma_{G_i}(v_i)$?

We say that the numbering is *good*, if it satisfies this condition. Problem 2 and problem 2′ are indeed equivalent. Suppose G is such that every subgraph contains a P_4-free vertex. To find a good numbering, we choose for v_n a P_4-free vertex of G, then we choose for v_{n-1} a P_4-free vertex of $G \setminus \{v_n\}$, more generally we choose for v_{n-i} a P_4-free vertex of $G \setminus \{v_n, \ldots, v_{n-i+1}\}$.

Conversely, suppose that $v_1, \ldots, v_n$ is a good numbering. Let H be a subgraph of G. We want to prove that H contains a P_4-free vertex. Let $i = \max\{j : v_j \in H\}$. Then $\Gamma_H(v_i) \subset \Gamma_{G_i}(v_i)$. Therefore v_i is a P_4-free vertex in H.

Algorithms that are linear in the number of edges have been designed to recognize P_4-free graphs (see [4]). Therefore Problem 2′ (and hence Problem 2) can be decided in $\mathcal{O}(n^4)$ time. Hence slightly triangulated graphs can be recognized in $\mathcal{O}(n^{4.376})$ time.

3 Finding a maximum clique in a slightly triangulated graph

Let $v_1, \ldots, v_n$ be a good numbering of a slightly triangulated graph G, and let G_i be the subgraph induced by $v_1, v_2, \ldots, v_i$. We will search for a maximum clique using the obvious fact that $\omega(G) = \max\{\omega(v_i \cup \Gamma_{G_i}(v_i)) : i \in [1, n]\}$. Therefore, our search for a maximum clique will be restricted to subgraphs which are P_4-free.

P_4-free graphs are also known as *cographs*. A cograph has a unique tree representation (called a cotree) which corresponds to the recursive decomposition of the cograph in connected components in the cograph and its complement. The existence (and uniqueness) of such a decomposition is a straightforward consequence of the following lemma due to Seinsche [11].

Lemma 1 (Seinsche) *If H is a cograph then H or $\overline{H}$ is disconnected.*

The cotree T of a cograph H is defined recursively by:

- If H is just one vertex, then the cotree of H is isomorphic to this vertex.
- If H is disconnected, let $A_1, \ldots, A_k$ be the connected components of H. Then the root of the cotree T of H is labelled '0' and the subtrees at the root are the cotrees of $A_1, \ldots, A_k$.
- If H is connected, let $A_1, \ldots, A_k$ be the connected components of $\overline{H}$. Then the root of the cotree T of H is labelled '1' and the subtrees at the root are the cotrees of $A_1, \ldots, A_k$

The most efficient algorithms [4] for the recognition of cographs and the construction of the corresponding cotrees (complexity $\mathcal{O}(m+n)$) are incremental in the sense that the vertices are processed one at a time (given a cograph $H \cup v$ and T the cotree of H, the cotree of $H \cup v$ is obtained by modifying T).

Once the cotree T of a cograph H is constructed, the computation of $\omega(H)$ is done recursively using the relation:

$$\omega(H[r;T_1,\ldots,T_k]) = \begin{cases} \sum_{i=1}^{k} \omega(H[T_i]) & \text{if } r=1 \\ \max_{i=1,\ldots,k} \omega(H[T_i]) & \text{if } r=0. \end{cases}$$

Here $H[r;T_1,\ldots,T_k]$ represents the cograph whose cotree $[r;T_1,\ldots,T_k]$ has r as root, and $T_1,\ldots,T_k$ as subtrees at the root. The computation of $\omega(H[r;T_1,\ldots,T_k])$ can be adapted to find a maximum clique. Let $Q_1,\ldots,Q_k$ denote the maximum cliques of the subgraphs of H whose cotrees are respectively $T_1,\ldots,T_k$. If $r=1$ then $Q_1 \cup Q_2 \ldots \cup Q_k$ is a maximum clique of $H[r;T_1,\ldots,T_k]$. Otherwise $r=0$ and a largest clique among $Q_1,\ldots,Q_k$ is a a maximum clique of $H[r;T_1,\ldots,T_k]$.

To sum up, to find $\omega(G)$, compute $\omega(v_i \cup \Gamma_{G_i}(v_i))$ for all i, using the algorithm above, and take the maximum. The overall complexity is $O(n^3)$.

4 A colouring algorithm

Let G be a slightly triangulated graph with a good numbering $v_1,\ldots,v_n$ of its vertices. As before, we denote by G_i the subgraph induced by $v_1,v_2,\ldots,v_i$. We denote by H_i the neighbourhood $\Gamma_{G_i}(v_i)$.

We will colour the vertices of G in the good numbering order. Let $S_1,\ldots,S_{\omega(G_{i-1})}$ be the colour classes of a perfect colouring of G_{i-1}. That is, the number of colours used in this colouring is equal to the clique number. From this perfect colouring of G_{i-1}, we will obtain a perfect colouring of G_i. But first, we need some preparatory lemmas.

4.1 Some lemmas for the colouring algorithm

The star-cutset lemma plays a key role in colouring perfect graphs. A *star-cutset* is a cutset C such that some vertex in C is adjacent to all the remaining vertices in C. Chvátal has shown in [3] that a minimal imperfect graph cannot have a star-cutset. The algorithmic proof of this lemma will be used in our colouring algorithm.

Lemma 2 (Chvátal) *If every proper subgraph of a graph G has a perfect colouring and if G has a star-cutset, then G admits a perfect colouring.*

Proof : Let G be such a graph, and let v be the center of the star-cutset (the vertex which dominates C). Let $A_1,A_2,\ldots,A_k$ be the components of $G \setminus C$. As every proper subgraph of G is ω-colourable, the subgraphs $G \setminus A_1$ and $C \cup A_1$ are ω-colourable. Let B_1 be the colour class of v in $G \setminus A_1$, and B_2 the colour class of v in $C \cup A_1$. Then $B_1 \cup B_2$ is a stable set, which meets (has a non-empty intersection with) every maximum clique of G. The graph $G \setminus (B_1 \cup B_2)$ is $\omega(G)-1$ colourable. This colouring of $G \setminus (B_1 \cup B_2)$ can be completed with $B_1 \cup B_2$ into a perfect colouring of G.
□

Our colouring algorithm will examine the neighbourhood H_i to find an available colour for v_i, or a star-cutset, or a colour class that intersects all the maximum cliques of G_i.

In the remainder of this subsection we treat the case of the colouring algorithm when the $(\omega - 1)$-cliques of H_i are not all in the same component. The following lemmas will show that in this case, either we can colour v_i without changing the colouring of G_{i-1} or we can find a star-cutset in G_i.

Let $N_1 \cup N_2$ be a partition of H_i such that

$$\forall n_1, n_2 \in N_1 \times N_2, n_1 n_2 \notin E(G_i)$$

and such that there are $(\omega - 1)$-cliques in both N_1 and N_2. Let I denote the set of vertices of $G_i \setminus (v_i \cup H_i)$ adjacent to both N_1 and N_2. i.e.

$$I = (\Gamma_{G_i}(N_1) \cap \Gamma_{G_i}(N_2)) \setminus v_i.$$

Lemma 3 *If $I = \emptyset$ then $\{v_i\}$ is a cutset.*

Proof : Suppose that $I = \emptyset$. If v_i is not a cutset there exists a chordless path from N_1 to N_2. But this would imply that G_i contains a chordless cycle with 5 or more vertices, which is impossible. □

Lemma 4 *If $\exists x \in I, N_1 \subset \Gamma_{G_i}(x)$ or $N_2 \subset \Gamma_{G_i}(x)$, then v_i is the center of a star-cutset, or can receive the colour of x.*

Proof : Suppose that $\exists x \in I, N_1 \subset \Gamma_{G_i}(x)$ or $N_2 \subset \Gamma_{G_i}(x)$. Without loss of generality we assume that $N_2 \subset \Gamma_{G_i}(x)$. Let x be such that $|\Gamma_{G_i}(x) \cap H_i|$ is a maximum among the vertices of I that satisfy $N_2 \subset \Gamma_{G_i}(x)$. If $N_1 \subset \Gamma_{G_i}(x)$ then v_i can receive the colour of x. So assume that $N_1 \not\subset \Gamma_{G_i}(x)$. If $v_i \cup (\Gamma_{G_i}(x) \cap H_i)$ is not a star-cutset, then there is a chordless chain included in $G_i \setminus (\Gamma_{G_i}(x) \cap H_i))$ whose length is at least 2, and joining x to $N_1 \setminus \Gamma_{G_i}(x)$. Let $y_1, y_2, \ldots, y_k = x$ be this chain. We have $k \geq 3$, $y_1 \in N_1$, and $\forall j \geq 2, y_j \notin \Gamma_{G_i}(v_i)$. We will consider two cases. We abbreviate $\Gamma_{G_i}(z) \cap N_j$ by $N_j(z)$.
case 1 Suppose that $N_2 \setminus N_2(y_2) \neq \emptyset$. Let z be a vertex of $N_2 \setminus N_2(y_2)$. Let l be the smallest index such that $y_l z \in E$. Then $\{v_i, y_1, y_2, \ldots, y_l, z\}$ induces a chordless cycle with length at least 5, a contradiction.
case 2 Suppose that $N_2(y_2) = N_2$. Then k is at least 4. Otherwise there would be a $(\omega + 1)$-clique in $x \cup y_2 \cup N_2$. As we chose x such that $|\Gamma_{G_i}(x) \cap H_i|$ is a maximum and as $N_1(y_2) \setminus N_1(x) \neq \emptyset$, $\exists t \in N_1(x) \setminus N_1(y_2)$. If $ty_1 \in E$ then $\{y_1, t, x, n_2, y_2\} = C_5$, where n_2 is any vertex of N_2. If $ty_1 \notin E$ then let l be the smallest index such that $y_l t \in E$. Then l is at least 3 and $\{v_i, y_1, y_2, \ldots, y_l, t\}$ induces a chordless cycle with length at least 5. □

Lemma 5 *If $\forall x \in I, N_1 \not\subset \Gamma_{G_i}(x)$ and $N_2 \not\subset \Gamma_{G_i}(x)$ then v_i is the center of a star-cutset.*

Proof : We assume that the intersection I is nonempty and that $\forall x \in I, N_1 \not\subset \Gamma_{G_i}(x)$ and $N_2 \not\subset \Gamma_{G_i}(x)$. Consider $x \in I$ such that $|\Gamma_{G_i}(x) \cap H_i|$ is a maximum. We have $N_1 \not\subset \Gamma_{G_i}(x)$ and $N_2 \not\subset \Gamma_{G_i}(x)$. If $v_i \cup N_1(x) \cup N_2$ is not a star-cutset then let $x_1, x_2, \ldots, x_k = x$ be a chordless chain joining $N_1 \setminus \Gamma_{G_i}(x)$ to x in $G_i \setminus (v_i \cup N_1(x) \cup N_2)$. We have $k \geq 3$, $x_1 \in N_1$, and $\forall j \geq 2, x_j \notin H_i$. We have $N_2(x) \subset N_2(x_2)$. Otherwise let $z \in N_2(x) \setminus N_2(x_2)$ and let l be the smallest index such that $x_l z \in E$. Then $\{v_i, x_1, x_2, \ldots, x_l, z\}$ induces a chordless cycle with length at least 5, a contradiction. Therefore we indeed have $N_2(x) \subset N_2(x_2)$. Since $|\Gamma_{G_i}(x) \cap H_i|$ is maximum there exists $u \in N_1(x) \setminus N_1(x_2)$. We have $x_1 u \in E$. Otherwise there exists l such that $\{v_i, x_1, x_2, \ldots, x_l, u\}$ induces a chordless cycle with length at least 5, a contradiction. If $k \geq 4$ then $x x_2 \notin E$ and $\{x_1, u, x, n_2, x_2\} = C_5$, where $n_2 \in N_2(x)$. To sum up, there is a chordless chain joining x to $N_1 \setminus N_1(x)$ whose length equals 2 and with the properties found above on the inclusions of the neighbourhoods. The same holds for $N_2 \setminus N_2(x)$. Let $\{x, b_1, a_1\}$ be a chordless chain from x to $N_1 \setminus N_1(x)$, and let $\{x, b_2, a_2\}$ be a chordless chain from x to $N_2 \setminus N_2(x)$. As $N_2(x) \subset N_2(b_1)$ and $N_2(x) \setminus N_2(b_2) \neq \emptyset$, we must have $b_1 \neq b_2$. Otherwise $|\Gamma_{G_i}(x) \cap H_i|$ would not be maximum. Let $u_1 \in N_1(x) \setminus N_1(b_1)$, and let $u_2 \in N_2(x) \setminus N_2(b_2)$. The vertices u_j and a_j are not adjacent ($j \in \{1, 2\}$). Otherwise $\{v_i, a_j, b_j, x, u_j\}$ would be a C_5. To finish the proof of lemma 5 we have two cases to consider.

case 1 $a_1 b_2 \notin E$ and $a_2 b_1 \notin E$.

case 1.1 If $b_1 b_2 \in E$ then $\{v_i, a_1, b_1, b_2, a_2\} = C_5$.

case 1.2 If $b_1 b_2 \notin E$ then $\{v_i, a_1, b_1, x, b_2, a_2\} = C_6$.

case 2 If $a_1 b_2 \in E$ then $\{v_i, a_1, b_2, x, u_2\} = C_5$. Similarly, if $a_2 b_1 \in E$ then $\{v_i, a_2, b_1, x, u_1\} = C_5$. □

4.2 The colouring algorithm

The algorithm proceeds by recursion on the number of vertices of G_i. If we find a star-cutset in G_i or a colour class of G_{i-1} that intersects all the maximum cliques of G_i, then we will be able to colour G_i recursively. Let $S_1, \ldots, S_{\omega(G_{i-1})}$ denote the colour classes of a perfect colouring of G_{i-1}. We are going to find a perfect colouring for G_i by examining the colouring of H_i in G_{i-1}.

Begin (Colouring Algorithm)

case 1 $\omega(G_i) = 2$.
Then G_i is bipartite, therefore easily colourable.

case 2 $\omega(H_i) = \omega(G_{i-1})$.
We introduce a new colour for v_i.

case 3 $\omega(H_i) < \omega(G_i) - 1$.
Then $\chi(G_i \setminus S_1) = \omega(G_i) - 1$. Recursively we colour $G_i \setminus S_1$ with $\omega(G_i) - 1$ colours. The union of the colour classes of $G_i \setminus S_1$ together with S_1 yields a perfect colouring of G_i.

case 4 $\omega(H_i) = \omega(G_i) - 1$.
We can suppose that all the $\omega(G_i)$ colours appear in H_i, otherwise we use the missing

colour for v_i.

case 4.1 H_i is connected.

By Seinsche's lemma, the subgraph H_i admits a partition into two parts A and B, such that $\forall a, b \in A \times B, ab \in E(G_i)$. Note that a colour cannot be simultaneously present in A and B, and that we have $\omega(A) + \omega(B) = \omega(G_i) - 1$. Without loss of generality, suppose that the colours $[1, \ldots, p]$ appear in A, and that the colours $[p+1, \ldots, p+q]$ appear in B. We have $\omega(A) \leq p - 1$ or $\omega(B) \leq q - 1$, otherwise we would have $\omega(G_i) = p+q = \omega(A)+\omega(B) = \omega(G_i)-1$, a contradiction. Without loss of generality suppose that $\omega(A) \leq p-1$. Recursively, we obtain a perfect colouring of $G_{i-1}[S_1, \ldots, S_p] \cup v_i$ with p colours. This new colouring together with $S_{p+1}, \ldots, S_{p+q}$ yield a perfect colouring of G_i.

case 4.2 H_i is disconnected.

Let $A_1, \ldots, A_k$ be the connected components of H_i.

case 4.2.1 The $(\omega - 1)$-cliques of H_i are not all in the same component.

case 4.2.1.1 $I = \emptyset$.

Then according to lemma 3, $\{v_i\}$ is a cutset.

case 4.2.1.2 $\exists x \in I, N_1 \subset \Gamma_{G_i}(x)$ or $N_2 \subset \Gamma_{G_i}(x)$.

Then lemma 4 shows that v_i is the center of a star-cutset (this star-cutset is characterized in lemma 4) or can receive the colour of x.

case 4.2.1.3 $\forall x \in I, N_1 \not\subset \Gamma_{G_i}(x)$ and $N_2 \not\subset \Gamma_{G_i}(x)$.

Then according to lemma 5, v_i is the center of a star-cutset (this star-cutset is characterized in the same lemma).

case 4.2.2 All the $(\omega - 1)$-cliques of H_i are in the same component.

Without loss of generality, suppose that this component is A_1. As A_1 is a connected P_4-free subgraph, A_1 admits a partition into two parts B and B' such that $\forall b, b' \in B \times B', bb' \in E(G_i)$. We have

$$\omega(A_1) = \omega(B) + \omega(B') = \omega(G_i) - 1.$$

Suppose without loss of generality that the colours $1, \ldots, p$ appear in B and that the colours $p+1, \ldots, p+q$ appear in B'. If $\omega(B) < p$ and $\omega(B') < q$ then

$$\omega(G_i) - 1 = \omega(A_1) = \omega(B) + \omega(B') \leq p + q - 2 = \omega(G_i) - 2.$$

This is a contradiction. Therefore $\omega(B) = p$ or $\omega(B') = q$. Suppose without loss of generality that $\omega(B) = p$. Let S_1 be a colour class of G_{i-1} which appears in B. The colour class S_1 meets every maximum clique of G_i. Therefore, $\omega(G_i \setminus S_1) = \omega(G_i) - 1$. We colour recursively $G_i \setminus S_1$ with $\omega(G_i) - 1$ colours. The union of the colour classes of $G_i \setminus S_1$ together with S_1 yields a perfect colouring of G_i.

End (Colouring Algorithm)

This colouring algorithm should be regarded as a constructive proof of the perfection of slightly triangulated graphs, which gives as a by-product a perfect colouring. From a practical point of view, it should be noted that the use of the star-cutset lemma leads to a complexity which is not polynomial in the worst cases.

Let us show that, in general, a colouring algorithm which uses the star-cutset lemma cannot be polynomial. Assume that a graph G has a star-cutset C. Let $A_1, A_2, \ldots, A_k$ be the connected components of $G \setminus C$. Let $\phi(G)$ denote the time spent to colour the graph G, and $\phi(n)$ the maximum time spent to colour a graph of order n. Then

$$\phi(C \cup A_1 \cup A_2 \ldots \cup A_k) = \phi(C \cup A_1) + \phi(C \cup A_2) + \cdots + \phi(C \cup A_k).$$

In the extreme case where $k = 2$ and $|A_1| = |A_2| = 1$, we have $\phi(C \cup A_1 \cup A_2) = \phi(C \cup A_1) + \phi(C \cup A_2)$. This implies that $\phi(n) = 2 \times \phi(n-1)$. In this worst case the complexity is exponential.

A more efficient colouring algorithm for slighly triangulated graphs would certainly use a new combinatorial characterization of these graphs. This has been the case for another class of perfect graphs also generalizing triangulated graphs (see [7] for the details of the story).

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