Efficient algorithms for reliability of a consecutive 2-out-of-r-from-n: F system

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Abstract

The previous literature on consecutive k-out-of-r-from-n: F systems gives recursive equations for the system reliability only for the special case when all component probabilities are equal. This paper deals with the problem of calculating the reliability for a (linear or circular) consecutive 2-out-of-r-from-n: F system with unequal component probabilities. New recursive equations are proposed, which have time complexity of $O(n)$ for the linear system, and $O(nr)$ for the circular system.

1. Introduction

Let S be a system which consists of n linearly or circularly ordered components where each component is either operational or failed. S is called a consecutive k-out-of-r-from-n: F system if the system fails whenever there are r consecutive components at least k of which are failed. It is a generalization of both k-out-of-n: F and consecutive k-out-of-n: F systems, in the sense that these two systems are obtained, when r=n and k=r, respectively. There are two cases, depending on whether the components are arranged on a straight line (linear system) or on a circle (circular system).

Australasian Journal of Combinatorics 14 (1996), pp. 31-36

Consecutive k-out-of-r-from-n: F systems were introduced by Griffith [1], and they arise in quality-control problems and inspection procedures, and in radar detection problems [2]. For the equal component probabilities, Sfakianakis et al. gave an explicit solution for k=2 and provided upper and lower bounds for $k \geq 3$, by using Bonferonni type inequalities [2]. Both approaches require considerable computational effort. Papastavridis et al. gave upper and lower bounds for the reliability of linear or circular system with unequal component probabilities [3].

A telecommunication system uses n-byte messages. The last bit of every byte is a parity bit (1 when the parity of the byte is correct). An error detector indicates an error when it finds 2 or more errors in a "window" of width 4 in the parity bit sequence. This is an example of a consecutive 2-out-of-4-from-n system.

For $k \geq 3$ an explicit solution is difficult to find, and there does not seem to exist any explicit solutions for a general system with unequal component probabilities. This paper considers both linear and circular consecutive 2-out-of-r-from-n: F systems with unequal component probabilities. We propose new recursive equations for the explicit solution of such systems. Furthermore, we present two new algorithms for the linear and circular systems which have time complexity of $O(n)$ and $O(nr)$, respectively.

2. Model

Assumptions

A. The system and each component are either operational or failed.

B. For given n, all components are mutually statistically independent.

C. The system fails if and only if within r consecutive components, there are at least k failed ones.

Notation

n : number of components in the system.

r : a "window" of r consecutive out of n components.

k : minimum number of failed components in the window of length r, which cause system failure.

i : index for a component; $i = 1, 2, \ldots, n$.

p_i : operational probability of component i.

q_i : $\equiv 1.0 - p_i$, failure probability of component i.

$E_m(i, j)$: event that there exists single failed component (component m) on the line between i and j.

$L(n,r,2)$: linear consecutive 2-out-of-r-from-n: F system, for $2 \le r \le n$. linear 2-out-of-n: F system, for $2 \le n \le r$.

$C(n,r,2)$: circular consecutive 2-out-of-r- from-n: F system, for $2 \le r \le n$.

$R_L(n,r,2)$: reliability of $L(n,r,2)$.

$R_C(n,r,2)$: reliability of $C(n,r,2)$.

3. Recursive Equation

3.1 Linear system

For p_n (the operational probability of component n), and q_n (the failure probability of component n) we have the following equation both for $2 \le n \le r$ and for $2 \le r \le n$:

$$\begin{aligned} R_L(n,r,2) &= p_n \cdot R_L(n-1,r,2) \\ &\quad + q_n \cdot \prod_{i=n-r+1}^{n-1} p_i \cdot R_L(n-r,r,2) \end{aligned} \tag{1}$$

with boundary condition :

$R_L(j,r,2) = 1.0$, for $j \le 1$, and $p_j = 1.0$, for $j \le 0$.

First compute $P(j) \equiv \prod_{i=j-r+1}^{j-1} p_i$ for every $j = 2,3,\ldots,n$. This requires $O(n+r) = O(n)$ time by first computing $O(r)$ and then computing $P(j+1) = P(j) \cdot p_j/p_{j-r+1}$ for each $j = 2,3,\ldots,n-1$. Once this is done, $R_L(n,r,2)$ can be computed in constant time. Since there are n such $R_L(n,r,2)$'s to compute, we need another $O(n)$ time. Therefore the total time required is $O(n)$.

To illustrate the computation procedure of the recursive formula Eq.(1), we consider $L(7,3,2)$. Starting with the boundary condition that $R_L(1,3,2) = R_L(0,3,2) = 1.0$ and $R_L(j,3,2) = 1.0$ for $j < 0$, we successively compute $R_L(2,3,2)$, $R_L(3,3,2)$, $\ldots$, $R_L(7,3,2)$ as follows:

$$R_L(2,3,2) = p_2 + q_2 p_1 \tag{2}$$
$$R_L(3,3,2) = p_3 \cdot R_L(2,3,2) + q_3 p_2 p_1 \tag{3}$$
$$R_L(4,3,2) = p_4 \cdot R_L(3,3,2) + q_4 p_3 p_2 \tag{4}$$
$$R_L(5,3,2) = p_5 \cdot R_L(4,3,2) + q_5 p_4 p_3 \cdot R_L(2,3,2) \tag{5}$$
$$R_L(6,3,2) = p_6 \cdot R_L(5,3,2) + q_6 p_5 p_4 \cdot R_L(3,3,2) \tag{6}$$
$$R_L(7,3,2) = p_7 \cdot R_L(6,3,2) + q_7 p_6 p_5 \cdot R_L(4,3,2) \tag{7}$$

Eq.(1) is a very simple recursive relation which needs only $r+1$ multiplications and one addition at each iteration. An algorithm Linear_system as shown in Fig.1 implements Eq.(1) and gives the exact system reliability. The algorithm is in terms of operational probabilities because using them usually causes less round-off errors for reasonably reliable systems. As with any numerical caluculation, care must be exercised about the maximum size of numbers and round-off errors.

The algorithm Linear_system requires a total of $5n$ multiplications/divisions and n additions. Bounds for $R_L(n, r, 2)$ are not needed since the computational effort to obtain exact system reliability is so small.

```
procedure  Linear_system(p_1 ~ p_n; R_L, n, r, 2)
  begin
    procedure  Linear(p_s ~ p_t; R, t, r)
      begin
        for  i := s + 1 - r until  s do
          set R(i) := 1.0;
          RR := p_s;
        for  i := s + 1 until  t do
          comment : p_m = 1.0, if m <= s - 1;
          begin
            R(i) := p_i · R(i - 1) + q_i · RR · R(i - r);
            RR := RR · p_i/p_{i-r+1}
          end
      end  Linear;
    Linear(p_1 ~ p_n; R, n, r);
    R_L := R(n)
  end  Linear_system;
```

Fig.1 Algorithm Linear_system.

3.2 Circular system

Suppose that the n components are labeled by the set $\{1, 2, \ldots, n\}$ in a clockwise rotation (component n is followed by component 1). Consider the $r-1$ consecutive components on the line between $n-r+2$ and n. If there are no failed components among them and a linear subsystem $L(n-r+1, r, 2)$ is operational, the circular system is operational. Since $E_{n-r+i+1}(n-r+2, n)$ implies that all the consecutive components from $n-r+2$ to $n-r+i$ and from $n-r+i+2$ to n are operational, the circular system is operational if it contains consecutive operational components from $n-2r+i+2$ to $n-r+1$ and from 1 to i, and a linear subsystem which consists of consecutive components on the line between $i+1$ and $n-2r+i+1$ is operational. Let $R_L(p_s \sim p_t; t, r, 2)$ be the reliability of linear subsystem consisting of consecutive components on the line between s and t; $R_L(n, r, 2)$ and $R_L(n-r, r, 2)$ mentioned above are the abbreviated expressions of $R_L(p_1 \sim p_n; n, r, 2)$ and $R_L(p_1 \sim p_{n-r}; n-r, r, 2)$, respectively. The $E_{n-r+i+1}$'s can be disjoint events, then we have the following reliability equation for $C(n,r,2)$ as:

$$
\begin{aligned}
R_C(n,r,2) \quad = \quad & \prod_{i=n-r+2}^{n} p_i \cdot R_L(p_1 \sim p_{n-r+1}; n-r+1, r, 2) \\
& + \sum_{i=1}^{r-1} \{(\prod_{j=1}^{i} p_j) \cdot (\prod_{j=n-2r+i+2}^{n-r+i} p_j) \\
& \qquad \cdot q_{n-r+i+1} \\
& \qquad \cdot (\prod_{j=n-r+i+2}^{n} p_j) \\
& \qquad \cdot R_L(p_{i+1} \sim p_{n-2r+i+1}; \\
& \qquad\qquad n-2r+i+1, r, 2)\}
\end{aligned}
\tag{8}
$$

Since $1 \le i \le r-1$, there are at most $r-1$ distinct products $\{(\prod_{j=1}^{i} p_j) \cdot (\prod_{j=n-2r+i+2}^{n-r+i} p_j) \cdot q_{n-r+i+1} \cdot (\prod_{j=n-r+i+2}^{n} p_j)\}$, which can be computed in $O(r)$ time. Furthermore, the reliability of the linear system can be computed in $O(n)$ time as shown before. Since there are at most r reliabilities of linear system to compute, the total time required is $O(r) + O(nr) = O(nr)$.

By the use of preceeding $O(n)$ algorithm Linear_system, we can construct an algorithm Circular_system of $O(nr)$ as shown in Fig.2.

The algorithm Circular_system requires a total of $5nr - 10r^2 + 20r - 11$ multiplications/divisions and $nr - 2r^2 + 3r - 1$ additions.

```
procedure  Circular_system(p_1 ~ p_n; R_C, n, r, 2)
   begin
      R_1 := p_{n-r+2};
      for  i := n-r+3 until  n do  R_1 := R_1 · p_i;
      Linear(p_1 ~ p_{n-r+1}; R, n-r+1, r);
      R_C := R_1 · R(n-r+1);
      R_2 := p_1;
      for  i := 2 until  r-1 do  R_2 := R_2 · p_i;
      R_3 := 1.0;
      for  i := 0 until  r-2 do
         begin
            R_1 := R_1 · p_{n-r-i+1}/p_{n-i};
            Linear(p_{r-i} ~ p_{n-r-i}; R, n-r-i, r);
            R_C := R_C + R_2 · R(n-r-i) · R_1 · q_{n-i} · R_3;
            R_2 := R_2/p_{r-i-1};
            R_3 := R_3 · p_{n-i}
         end
   end  Circular_system;
```

Fig.2 Algorithm Circular_system.

Acknowledgement

The first author would like to thank the members of the School of Mathematics and Statistics at Curtin University of Technology in Perth, Australia, for their hospitality during 1994-1995 when the work on this paper was carried out.

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(Received 25/5/95)