GRAPHS WITH EQUIDISTANT CENTRAL NODES

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ABSTRACT. An F-graph is a graph G whose center $C(G)$ contains at least two nodes, and the distance between any two nodes of $C(G)$ equals the radius of G. In this paper we study properties of these graphs, and describe some ways of constructing them.

INTRODUCTION

One of the problems frequently studied in the application of graphs is the problem of facility location in central nodes. Some emergency facilities located in the same region can interfere with others when responding to a particular emergency. As follows from the definition, the central nodes of F-graphs are separated as much as possible to minimize the interference between corresponding facilities ([3],[2]).

All the terminology as well as notation except for that given here is taken from [1]. By a graph we will understand a finite, undirected, connected (if not stated otherwise) graph without loops or multiple edges. The set of nodes of a graph G is denoted by $V(G)$, and the set of edges by $E(G)$. The distance between nodes x and y of a graph G, denoted by $d_G(x,y)$, is the least number of edges in an $x-y$ path in G. The eccentricity $e_G(x)$ is $max\{d_G(x,w)\}$ for all $w \in V(G)$. A node v for which $e_G(x) = d_G(x,v)$ is called an eccentric node for x. The radius $r(G)$ and diameter $d(G)$ are the minimum and maximum eccentricities, respectively. If G is a disconnected graph, then $d_G(x,y) = \infty$ if there is no path between nodes x and y in G and $r(G) = d(G) = \infty$. The neighborhood of a node $x \in V(G)$ is denoted by $N_G(x)$. The distance between a node $x \in V(G)$ and a nonempty subset J of $V(G)$ is the minimum of the distances $d_G(x,y)$, for every $y \in J$. The distance $d_G(J,K)$ between two nonempty subsets J,K of $V(G)$ is the minimum of $d_G(x,y)$, where $x \in J,\ y \in K$.

The center of a graph G, denoted by $C(G)$, consists of the nodes of minimum eccentricity. A node $x \in C(G)$ is a central node of G.

Let $F \equiv \{G \mid |C(G)| \geq 2;\ x,y \in C(G),\ x \neq y \to d_G(x,y) = r(G)\}$. A graph $G \in F$ will be called an F-graph. Obviously, every complete graph with at least two nodes is an F-graph with radius one. It is also the only selfcentric graph which is an F-graph. Thus we will consider only F-graphs with radius greater than or equal to two.

In paper [3] the notion of F-graph is introduced and some results are given on the existence of such graphs with prescribed radius and diameter. Also it is shown that any graph G can be embedded as an induced subgraph in a supergraph $H \in F$ with $d(H) = 4$, and $|V(H)| = |V(G)| + 5$.

In this paper we generalize some of the results of [3] and introduce some new results. Given an F-graph G with at least three central nodes, we show that the reduction $G - x$ for every central node x of G is also an F-graph. We give a necessary and sufficient condition for a graph G to be an F-graph, and introduce some constructions of F-graphs.

1. Properties of F-graphs.

Lemma 1.1. *Let G be an F-graph with $r(G) \geq 2$ and x be an arbitrary central node of G. Then there is at least one node $q \in V(G) - C(G)$ such that $d_G(x, q) = r(G)$.*

Proof. Suppose that $d_G(x, s) \leq r(G) - 1$ for every $s \in V(G) - C(G)$. Since $x \in C(G)$, the neighborhood of x in G is nonempty and the eccentricity of any node from $N_G(x)$ is less than or equal to $r(G)$, which is a contradiction.

Corollary 1.1. *Let G be a graph with the radius $r(G) \geq 2$. If there is a central node of G such that it has only one eccentric node, then G is not an F-graph.*

As shown in [3], a central node of any F-graph G cannot be a cutnode. If a central node x of an F-graph G was a cutnode, then there are at least two components in $G - x$ such that one of them contains a central node y different from x and the second contains at least one node z different from x and y. Then x lies on every $y - z$ geodesic in G, which is a contradiction.

Theorem 1.1 (Reducibility). *Let G be an F-graph with $|C(G)| \geq 3$, $r(G) = r \geq 2$, and let x be a central node of G. Then the graph $G - x$ is an F-graph with the same radius $r(G - x) = r(G)$ and the center of $G - x$ is $C(G - x) = C(G) - \{x\}$.*

Proof. Since x cannot be a cutnode of G, the graph $G - x$ is connected. The eccentricity of any node in $G - x$ is greater than or equal to its eccentricity in G.

a) Suppose that $y \in C(G) - \{x\}$. Then $e_G(y) = r(G)$. For any node $q \in V(G)$, $q \neq x$, the length of a $y - q$ geodesic P is less than or equal to $r(G)$. Since $d_G(y, x) = r(G)$, P cannot contain the node x. Thus $d_G(y, q) = d_{G-x}(y, q)$. For any central node y of G there are at least two nodes in G with distance $r(G)$ from y. (If $|C(G)| > 2$ then these two nodes are from $C(G)$ and if $|C(G)| = 2$ then the assertion follows from Lemma 1.1.) Hence there is at least one such node in $G - x$ too. Therefore $e_{G-x}(y) = r(G)$.

b) Let $y \in V(G - x)$, $y \notin C(G) - \{x\}$. Then $e_G(y) > r(G)$ and there is a node q different from x in G such that $d_G(y, q) > r(G)$. Then also $d_{G-x}(y, q) > r(G)$.

c) To prove that $G-x$ is an F-graph we must show that $d_{G-x}(y,z)=r$ for every two central nodes y,z from $C(G)-\{x\}$. No $y-z$ geodesic in G can contain the node x. Therefore, $d_G(y,z)=d_{G-x}(y,z)=r$.
This completes the proof.

Remark 1.1. *The converse assertion does not hold. The graph G in the Fig.1 is not an F-graph, but $G-x_9$ is an F-graph.*

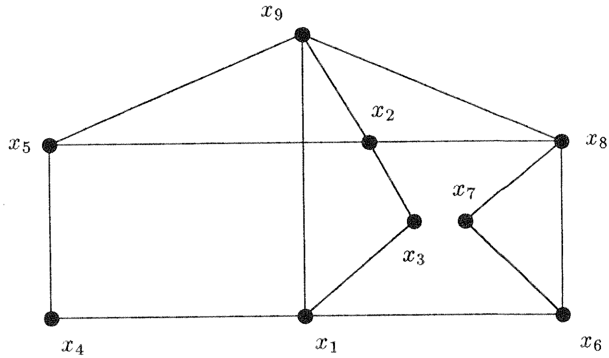

Fig.1

The following theorem gives a necessary and sufficient condition for a graph G to be an F-graph.

Theorem 1.2. *Let G be a graph with $r(G)=r\geq 2$ and $C(G)=\{x_1,x_2,\ldots,x_k\}$ $k\geq 3$. Then G is an F-graph if and only if for every $i=1,2,\ldots,k$ the graph $G-x_i$ is an F-graph with $r(G-x_i)=r$ and $C(G-x_i)=C(G)-\{x_i\}$.*

Proof. If G is an F-graph, then the assertions follow immediately by Theorem 1.1. Let $G-x_i$ be an F-graph, $r(G-x_i)=r$ and $C(G-x_i)=C(G)-\{x_i\}$ for $i=1,2,\ldots,k$. Let G not be an F-graph. Then there are two different nodes x_i,x_j from $C(G)$ such that $d_G(x_i,x_j)<r$. Since $|C(G)|\geq 3$, there is a node $x_m\in C(G)$ different from x_i and x_j. As follows from the assumption, $G-x_m$ is an F-graph, $x_i,x_j\in C(G)-x_m$ and $r(G-x_m)=r$. Thus $d_{G-x_m}(x_i,x_j)=r$. Every x_i-x_j geodesic in G must pass through the node x_m. Then there is an x_j-x_m path P in G of length less than r, which does not contain the node x_i. The graph $G-x_i$ is an F-graph, $r(G-x_i)=r$ and $x_j,x_m\in C(G-x_i)$. Then the distance $d_{G-x_i}(x_j,x_m)<r$, which is a contradiction.

In the next part of this paper we will examine the possibility of the extension of an F-graph G about one node. First we define the set of nodes of G such that adding a new node to $V(G)$, which is connected with all the nodes from this set, gives again an F-graph.

Definition 1.1. *Let G be a graph with $r(G)=r\geq 2$. Let J be a subset of $V(G)$, $|J|\geq 2$, with the following properties:*
1. $d_G(y,J)=r-1$ for every $y\in C(G)$

2. $d_G(t, J) \leq r - 1$ for every $t \notin C(G)$
3. for every node $u \in V(G) - C(G)$ there is at least one node v such that $d_G(u, v) > r$ and $d_G(u, J) + d_G(v, J) \geq r - 1$.

We will call such a set J an e-set of G.

Theorem 1.3. *The neighborhood $N_G(x)$ of any central node of an F-graph G is an e-set of $G - x$.*

Proof.
1) Since G is an F-graph, $d_G(x, y) = r$ for every $y \in C(G), y \neq x$. Therefore, $d_{G-x}(y, N_G(x)) = r - 1$.
2) $d_{G-x}(t, N_G(x)) \leq r - 1$ for every node $t \in V(G) - C(G)$, since $x \in C(G)$.
3) For every node $u \in V(G) - C(G)$ there is at least one node $v \in V(G) - C(G)$, $v \neq x$, such that $d_G(u, v) > r$. Then also $d_{G-x}(u, v) > r$ and $d_{G-x}(u, N_G(x)) + d_{G-x}(v, N_G(x)) \geq r - 1$.

Corollary 1.2. *Let G be an F-graph with $r(G) = 2$. Let $x \in C(G)$. Then $N_G(x)$ is an e-set in both G and $G - x$.*

Definition 1.2. *Let G be a graph. Let J be any subset of $V(G)$ and $x \notin V(G)$. Then the graph with the set of nodes $V(G) \cup \{x\}$ and the set of edges $E(G) \cup \{xy \mid y \in J\}$ is denoted by $(G + x)_J$.*

Theorem 1.4. *Let G be an F-graph with $r(G) = r \geq 2$ and $x \notin V(G)$. Let J be an e-set of G. Then $(G + x)_J$ is an F-graph, $r((G + x)_J) = r$ and $C((G + x)_J) = C(G) \cup \{x\}$.*

Proof. As follows from the properties of J, the eccentricity of x in $(G + x)_J$ is equal to r and $d_{(G+x)_J}(x, y) = r$ for every $y \in C(G)$. Since for any $y, z \in C(G)$, $y \neq z$, the shortest $y - z$ path in $(G + x)_J$ cannot contain the node x, the distance $d_{(G+x)_J}(y, z) = r$. Let $u \in V(G) - C(G)$. Then there is a node $v \in V(G) - C(G)$ such that $d_G(u, v) > r$ and $d_G(u, J) + d_G(v, J) \geq r - 1$, i.e. the eccentricity of u in $(G + x)_J$ is greater than r. This completes the proof.

Corollary 1.3. *Let G be an F-graph with $r(G) = 2$, $x \notin V(G)$ and $y \in C(G)$. Then the graph $(G + x)_{N_G(y)}$ is also an F-graph with radius two.*

Corollary 1.4. *Let G be an F-graph with $r(G) \geq 2$ and $|C(G)| = k$, $k \geq 3$. Then G is an $(H + x)_J$ graph for some F-graph H with $r(H) = r(G)$ and $C(H) = C(G) - \{x\}$. $N_G(x) = J$ is an e-set in H.*

Theorem 1.5. *Let G be a graph, $r(G) = r \geq 2$. Let $C(G) = \{x_1, x_2, \ldots, x_k\}$, $k \geq 2$. Then G is an F-graph if and only if $N_G(x_i)$ for $i = 1, 2, \ldots, k$ satisfy the following conditions in the (not necessarily connected) graph $H = G - C(G)$:*

1) $d_H(N_G(x_i), N_G(x_j)) = r - 2$ for every $i \neq j$, $i, j = 1, 2, \ldots, k$.
2) $d_H(u, N_G(x_i)) \leq r - 1$ for every $u \in V(H)$, $i = 1, 2, \ldots, k$.
3) For every $u \in V(H)$ there is a node $v \in V(H)$ such that $d_H(u, v) > r$ and $d_H(u, N_G(x_i)) + d_H(v, N_G(x_i)) \geq r - 1$ for $i = 1, 2, \ldots, k$.

Proof. Let G be an F-graph.

1) Let $i, j \in \langle 1, k\rangle$, $i \neq j$. Then $d_G(x_i, x_j) = r$ and there is an $x_i - x_j$ path P of length r in G, which does not pass through any other node of $C(G)$. Therefore $d_H(N_G(x_i), N_G(x_j)) = r - 2$.
2) Let $u \in V(H)$ and $x_i \in C(G)$ be two nodes for which $d_H(u, N_G(x_i)) > r - 1$. Then also $d_G(u, N_G(x_i)) > r - 1$, a contradiction.
3) Let $u \in V(H)$. Since $u \notin C(G)$ the eccentricity of u in G is $e_G(u) > r$ and hence there is a node $v \in V(H)$ such that $d_G(u, v) > r$. If $d_H(u, v) \leq r$, then there is an $u - v$ path P of length less than or equal to r in H. Since any $u - v$ path from H also exists in G, we arrive at a contradiction. Therefore $d_H(u, v) > r$. Suppose there is a node $x_i \in C(G)$ with $d_H(u, N_G(x_i)) + d_H(v, N_G(x_i)) < r - 1$. Then also $d_G(u, N_G(x_i)) + d_G(v, N_G(x_i)) < r - 1$ and $d_G(u, v) \leq r$, a contradiction.

Let H be a graph and let the subsets $J_1,\ J_2, \ldots, J_k$ satisfy the conditions of the theorem. We prove that the graph G formed by adding k new nodes $x_1, x_2, \ldots, x_k$ to H in such a way that $N_G(x_i) = J_i$, $i = 1, 2, \ldots, k$, is an F-graph with $C(G) = \{x_1, x_2, \ldots, x_k\}$.

1) Since $d_H(N_G(x_i), N_G(x_j)) = r - 2$, then $d_G(x_i, x_j) = r$ for every $i, j \in \langle 1, k\rangle$, $i \neq j$.
2) $d_G(x_i, u) \leq r$ for every $u \in V(H)$, as $d_H(u, N_G(x_i)) \leq r - 1$.
3) Let $u \in V(G) - C(G)$. Then $u \in V(H)$. There is a node v such that $d_H(u, v) > r$. If $d_G(u, v) \leq r$ then any $u - v$ geodesic of G contains a node $x_i \in C(G)$, $i \in \langle 1, k\rangle$. Then $d_H(u, N_G(x_i)) + d_H(v, N_G(x_i)) < r - 1$, which is a contradiction. Thus $e_G(u) > r$.

Corollary 1.5. *Let G be a graph, $r(G) = 2$. Let $C(G) = \{x_1, x_2, \ldots, x_k\}$, $k \geq 2$ be the center of G. Let $H = G - C(G)$. Then G is an F-graph if and only if*

1) $N_G(x_i) \cap N_G(x_j) \neq \varnothing$ for every $i, j = 1, 2, \ldots, k$; $i \neq j$.
2) $N_G(x_1), N_G(x_2), \ldots, N_G(x_k)$ are dominating sets of H.
3) For every node $u \in N_G(x_i)$ there is at least one node $v \notin N_G(x_i)$ with $d_H(u, v) \geq 3$.

Theorem 1.6. *Let G be an F-graph with $r(G) = r \geq 2$ and $|C(G)| \geq 4$. Let P_1, P_2 be two paths of length r between two disjoint pairs of central nodes of G. Then*

1) P_1, P_2 are disjoint if r is odd;
2) P_1, P_2 have at most one common node if n is even.

Proof. Let x, y and z, t be two disjoint pairs of central nodes of G. Let P_1, P_2 be two paths of length r between x, y and z, t, respectively. Let q be a common node of P_1, P_2. If r is even, then q must be the node in the center of P_1 and P_2. If r is odd, then the distance between at least two of the nodes x, y, z, t is less than r, which is a contradiction.

2. Constructions of F-graphs.

Theorem 2.1. *Let G be a connected graph with $r(G) = r \geq 2$. Let n, k be natural numbers $r \leq n \leq d(G)$, $k \geq 2$, and let $X = \{x_1, x_2, \dots, x_k\}$ be an arbitrary subset of $V(G) = \{x_i\}_{i=1}^{|V(G)|}$ such that*

1) $d_G(x_i, x_j) = n$ for every $x_i, x_j \in X$, $i \neq j$;
2) $d_G(x_i, x_j) \leq n$ for every $x_j \in V(G) - X$ and $x_i \in X$.

Then there is an F-graph H with $r(H) = n$, $C(H) \equiv X$ containing G as an induced subgraph.

Proof. We construct the graph H as follows (Fig.2): Add four new nodes p_1, p_2, t_1, t_2 to $V(G)$. Each pair of nodes x_i, p_j for $i = 1, 2, \dots, k$; $j = 1, 2$ is connected by an $x_i - p_j$ path of length $n/2$ if n is even, $\lfloor n/2 \rfloor + 1$ if n is odd and every two such paths have in common at most one extremity. Finally we add two disjoint paths $p_1 - t_1$ and $p_2 - t_2$ each of length $\lfloor n/2 \rfloor$.

As follows from the construction, G is an induced subgraph of H. The eccentricity of any node from X in H is equal to n. The eccentricity of any other node in H is greater than n. Moreover, $d_H(x_i, x_j) = n$ for every $i, j = 1, 2, \dots, k$; $i \neq j$. Therefore H is an F-graph and $C(H) \equiv X$.

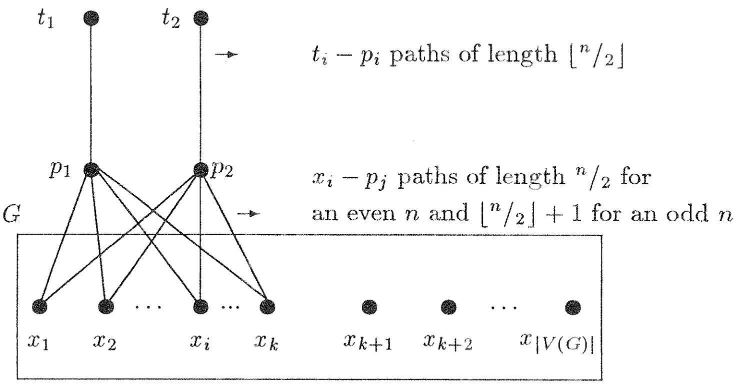

Fig.2

Remark 2.1. *For any graph G there is at least one such subset X for $n = d(G)$.*

As is shown in [3], for any graph G of order p there is an F-graph H with diameter 4 and $|V(H)| = p + 5$, such that G is an induced subgraph of H. Let G be an F-graph. The following construction shows that the substitution of any node $x \in V(G) - C(G)$ by an arbitrary graph results in an F-graph with radius $r(G)$ and center $C(G)$.

Let Q be a graph. Let G be an F-graph with $r(G) = r \geq 2$ and let s be an arbitrary node from $V(G) - C(G)$. We say that the graph H arose from G by the substitution of s by Q if

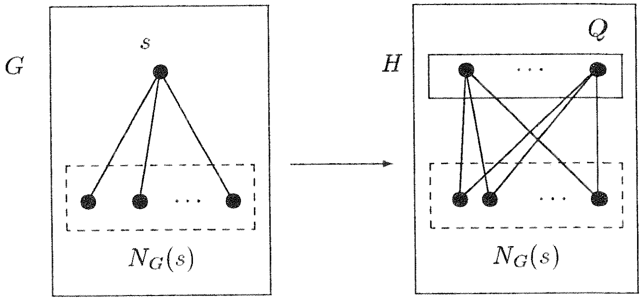

Fig. 3

$V(H) = V(G) \cup V(Q) - \{s\}$ and
$E(H) = E(G) \cup E(Q) \cup \{xy \mid x \in V(Q) \text{ and } sy \in E(G)\} - \{sy \mid sy \in E(G)\}$.

Theorem 2.2. *Let G be an F-graph with $r(G) = r \geq 2$. Let s be an arbitrary node from $V(G) - C(G)$. Let Q be a graph and H be the graph constructed from G by the substitution of s by Q in G. Then H is an F-graph with $r(H) = r(G)$ and $C(H) \equiv C(G)$.*

Proof. Let $x \in V(Q)$. Then $e_H(x) = e_G(s) > r$. Let $x \in V(G)$, $x \neq s$. Since $d_H(x,y) = d_G(x,s)$ for every $y \in V(Q)$, the eccentricity $e_H(x) = e_G(x)$. Let $x, y \in C(G)$. If $d_H(x,y) < r$, then there is an $x - y$ path P in H of length less than r. Any such path must contain at least one node from $V(Q)$. Replacing all nodes from $V(Q)$ by s in P gives an $x - y$ trail of length less than r in G, which is a contradiction. Thus $d_H(x,y) = r$.

Corollary 2.1. *Let G be an F-graph with $r(G) = r \geq 2$. Let $x \in V(G) - C(G)$ and $y \notin V(G)$. Then $(G + y)_{N_G(x)}$ is an F-graph with radius r and center $C(G)$.*

Let G_1, G_2 be two F-graphs with $r(G_1) = r(G_2) = r \geq 2$. Let $|C(G_1)| = |C(G_2)| = k$, where $k \geq 2$ is a natural number.

Let $C(G_1) = \{x_1, x_2, \ldots, x_k\}$; $C(G_2) = \{y_1, y_2, \ldots, y_k\}$. Let H be the graph constructed from the graph $G_1 \cup G_2$ by the removal of all the nodes y_i, for $i = 1, 2, \ldots, k$ and by adding the set of edges $\{x_i q \mid y_i q \in E(G_2)\}$, (Fig.4). We say that H is an amalgamation of G_1 and G_2 on $C(G_1)$ and $C(G_2)$. (Fig.4).

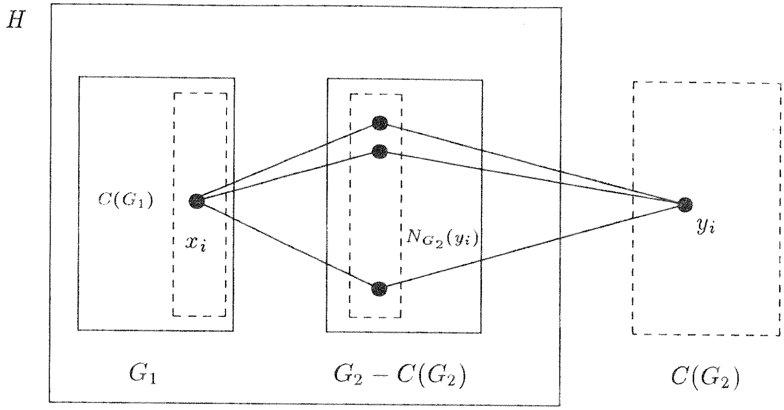

Fig.4

Theorem 2.3. *Let G_1, G_2 be two F-graphs with $r(G_1) = r(G_2) = r \geq 2$ and $|C(G_1)| = |C(G_2)| = k$, where $k \geq 2$ is a natural number. Then the amalgamation H of G_1 and G_2 on $C(G_1)$ and $C(G_2)$ is an F-graph with $r(H) = r$ and $C(H) = C(G_1)$.*

Proof.

1) $d_H(x_i, x_j) = d_{G_1}(x_i, x_j) = d_{G_2}(y_i, y_j) = r$ for every $i, j = 1, 2, \ldots, k;\ \ i \neq j$.
2) $d_H(x_i, q) = d_{G_1}(x_i, q) \leq r$ if $q \in V(G_1)$; for $i = 1, 2, \ldots, k$;
   $d_H(x_i, q) = d_{G_2}(y_i, q) \leq r$ if $q \in V(G_2)$ for $i = 1, 2, \ldots, k$.
3) Let $q \in V(G_1)$, $q \neq x_i$, for $i = 1, 2, \ldots, k$. Then $e_H(q) \geq e_{G_1}(q) > r$. Let $q \in V(G_2)$, $q \neq y_i$, for $i = 1, 2, \ldots, k$. Then $e_H(q) \geq e_{G_2}(q) > r$.

Therefore $C(H) = \{x_1, x_2, \ldots, x_k\}$, $r(H) = r$ and $d_H(x_i, x_j) = r$ for every $i, j = 1, 2, \ldots, k; i \neq j$.

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