

STRONGLY SQUARE-FREE STRINGS ON THREE LETTERS

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Abstract

A string is strongly square-free if it contains no Abelian squares; that is, adjacent substrings which are permutations of each other. We construct all 117 strongly square-free finite strings on an alphabet of three letters.

1 Introduction

An ordered sequence $x = x_1x_2 \cdots x_n$ of elements chosen from a fixed finite set, A , of distinct elements is called a *string* of length $|x| = n$ over the alphabet A . The empty string is denoted by λ . The ordered sequence $x_i x_{i+1} \cdots x_j$ is a *substring* of x if $1 \leq i \leq j \leq n$. In the interests of notational convenience and without loss of generality, we choose $A = \{0, \dots, n-1\}$ for fixed $n > 1$ as the alphabet. Every element of the alphabet is also a string. For each $a \in A$ we define a function $|x|_a$ to be the number of times that a appears in the string x . We freely concatenate strings and write the concatenation of strings x and y as simply xy . If a string $x = uv$ is the concatenation of two strings u and v then u is said to be a *prefix* of x and v is said to be a *suffix*. If $v \neq \lambda$ then u is said to be a *proper prefix* of x . Similarly, if $u \neq \lambda$ then v is said to be a *proper suffix* of x . If S_0, S_1 are sets of strings and x is a fixed string then we use the notation

$$S_0 x S_1 = \{uxv \mid u \in S_0, v \in S_1\}.$$

A string which consists of the concatenation of two equal substrings is called a *square*. A string without any substrings which are squares is said to be *square-free*.

For example, over the alphabet $\{0, 1\}$, 010010 is a square which contains another square 00 and it is a substring of 10100101. The strings 010 and 101 are square-free and, moreover, they cannot be extended by concatenation on either the right or the left without creating a square. Thue [14] observed that every two-sided infinite square-free string x over $\{0, 1, 2\}$ is a product of strings from the set:

$$01 \ 012 \ 0121 \ 02 \ 021 \ 0212.$$

It has been shown by Ross and Winkleman [12] that over any alphabet of at least three elements, the set of strings containing "squares" is not context-free.

An *Abelian square* is the concatenation of a string with a permutation of itself. Over the alphabet $\{0, 1\}$, 010100 is an Abelian square which contains the squares 0101, 1010, and 00. Since every square is an Abelian square, 010100 contains 4 Abelian squares. More formally:

Definition 1 *An Abelian square is a non-empty string of the form*

$$BB^\sigma = b_1 \cdots b_k b_{\sigma(1)} \cdots b_{\sigma(k)}$$

where σ is a permutation of $\{1, \dots, k\}$. A string is said to be strongly square-free if it contains no Abelian squares.

Over the alphabet $A = \{0, 1, 2\}$, the string 012201 is an Abelian square but 0102010 does not contain Abelian squares and cannot be extended over the alphabet $A = \{0, 1, 2\}$ without introducing Abelian squares. Clearly every strongly square-free string is square-free. Main [10] has shown that, for every alphabet with at least 16 elements, the set of strings which contain Abelian squares is not context-free. In a different direction, Entringer, Jackson, and Schatz [7] proved that every infinite binary string has arbitrarily long Abelian squares. Dekking [2] has shown that there exist infinite binary strings in which no four adjacent substrings appear which are permutations of one another; i.e., no two Abelian squares are adjacent.

Apparently Erdős [8] first raised the question of the minimum alphabet size over which there exist infinite strings without Abelian squares. This was a variant of the corresponding problem for squares raised and solved by Thue [13] in 1906. In 1970 Pleasants [11] showed that there existed an infinite strongly square-free string on an alphabet of 5 elements. This result was recently sharpened by Keränen [9] who showed the same was true for an alphabet of 4 elements, with a computer-aided proof. It was shown in [5] that every coordinate sequence of a binary Gray code of order n with one integer in the sequence deleted is a strongly square-free string of length $2^n - 1$. The purpose of this paper is to prove there are 117 distinct strongly square-free strings on $\{0, 1, 2\}$ and explicitly construct them. Accepting the result by Keränen [9], the case of just three letters is seen to be important because it is the last case for which all strongly square-free strings are finite.

It is folklore that any strongly square-free string over $\{0, 1, 2\}$ has length ≤ 7 [1]. This can be established, say, by diligently constructing the tree of possible strongly square-strings starting with 0 and observing that starting with 1 or 2 would yield the same tree.

Clearly, 0, 1, and 2 are strongly square-free and 01 10 02 20 12 21 are the only strings of length 2 that are strongly square-free over $\{0, 1, 2\}$. It is easy to write down the strongly square-free strings of length 3 as the next step, but we refrain here because they will be listed as elements of larger sets. The strongly square-free strings of each length $1, \dots, 7$ over $\{0, 1, 2\}$ are listed together as an appendix.

2 Central Strongly Square-Free Strings

In this section we discuss a class of strongly square-free strings which are obtained recursively from strongly square-free strings over smaller alphabets.

Definition 2 A string is said to be central if it contains $a \in A$ such that $|x|_a = 1$. A set of strings S is said to be central if there exists $a \in A$ such that $|x|_a = 1$ for all $x \in S$.

It is easy to see that the only strongly square-free strings on the binary alphabet $\{0, 1\}$ are

$$S_{01} = \{0 \ 1 \ 01 \ 10 \ 010 \ 101\}.$$

Note that each of these strings is central. In the same way, we get complete sets of strongly central square-free strings:

$$S_{02} = \{0 \ 2 \ 02 \ 20 \ 020 \ 202\}$$

on $\{0, 2\}$ and

$$S_{12} = \{1 \ 2 \ 12 \ 21 \ 121 \ 212\}$$

on $\{1, 2\}$.

Lemma 1 will show the sets

$$S_{12}0S_{12} \quad S_{02}1S_{02} \quad S_{01}2S_{01}$$

are strongly square-free strings. All the strings in these sets have length at least 3 and each set contains 36 strings. However, there are strings that appear in more than one of the sets. In Lemma 2, we determine the intersections of these sets. Further, all central strings with length ≥ 3 over the alphabet $\{0, 1, 2\}$ are in one of these sets. The following lemma yields a general technique for constructing central strongly square-free strings on arbitrary alphabets.

Lemma 1 If ω and ω' are strongly square-free strings over an alphabet A and $a \in A$ does not appear in ω or ω' then $\omega a \omega'$ is strongly square-free.

Proof. Suppose $\omega a \omega'$ contains an Abelian square BB^σ , where σ can be any permutation of B . Then, BB^σ cannot be a substring of ω or ω' since they are strongly square-free. Therefore, either a occurs in B or a occurs in B^σ . But since B^σ is a permutation of B , a must appear in both B and B^σ , contradicting the assumption that a appears only once in $\omega a \omega'$. $\square$

Note that if $a \in \{0, 1, \dots, n-1\}$ does not appear in ω or ω' then both ω and ω' are necessarily strings over an alphabet of at most $n-1$ elements.

Lemma 2

$$\begin{aligned} S_{12}0S_{12} \cap S_{02}1S_{02} &= \{2012 \ 2102 \ 20212 \ 21202\} \\ S_{12}0S_{12} \cap S_{01}2S_{01} &= \{1201 \ 1021 \ 12101 \ 10121\} \\ S_{02}1S_{02} \cap S_{01}2S_{01} &= \{0210 \ 0120 \ 02010 \ 01020\}. \end{aligned}$$

Proof. The strings in these intersections are all strongly square-free by Lemma 1. Let $x \in S_{12}0S_{12} \cap S_{02}1S_{02}$. Then

$$x = u0v = y1z \quad (1)$$

for some strings $u, v \in S_{12}$ and $y, z \in S_{02}$. From (1), either $u=y$, or u is a prefix of y , or y is a prefix of u . But if $u=y$ then $0 = 1$, which is impossible.

If u is a proper prefix of y then u also is in S_{02} because it is easy to see that S_{02} is closed under the taking of prefixes. But $S_{12} \cap S_{02} = \{2\}$. Therefore, $u = 2$ and so y can only be one of 20 or 202 . If $y = 20$ then (1) becomes $x = 20v = 201z$ which implies that $v = 1z$. In the same way, if $y = 202$ then (1) becomes $x = 20v = 2021z$ which implies that $v = 21z$. In either case, z is a proper suffix of v . Since S_{12} is closed under the taking of suffixes we conclude that $z \in S_{12} \cap S_{02} = \{2\}$. Therefore, $z = 2$ and so v can only be one of 12 or 212 . It follows that

$$S_{12}0S_{12} \cap S_{02}1S_{02} = \{2012 \ 20212\}.$$

in this case.

If y is a proper prefix of u then a similiar argument shows that in this case

$$S_{12}0S_{12} \cap S_{02}1S_{02} = \{2102 \ 21202\}.$$

We conclude that

$$S_{12}0S_{12} \cap S_{02}1S_{02} = \{2012 \ 2102 \ 20212 \ 21202\}.$$

In the same way we establish that

$$S_{12}0S_{12} \cap S_{01}2S_{01} = \{1201 \ 1021 \ 12101 \ 10121\}$$

and

$$S_{02}1S_{02} \cap S_{01}2S_{01} = \{0210 \ 0120 \ 02010 \ 01020\}.$$

This completes the proof. $\square$

Note that necessarily

$$S_{12}0S_{12} \cap S_{02}1S_{02} \cap S_{01}2S_{01} = \lambda.$$

Hence the union of these three sets contains exactly 96 distinct strings.

3 Non-Central Strongly Square-Free Strings

If x is any non-central strongly square-free string then $|x|_a \geq 2$ for all $a \in A$. Further, for any non-central strongly square-free string x , $|x| \geq 2|A|$. In particular, if x is a non-central strongly square-free string over $\{0, 1, 2\}$ then $|x| \geq 6$.

Lemma 3 *There are 6 non-central strongly square-free strings of length 6 over $\{0, 1, 2\}$.*

Proof. If $|x| = 6$ and x is non-central then x is a permutation of 001122 because every letter of $\{0, 1, 2\}$ must appear at least twice. Let $x = x_1 \cdots x_6$. Suppose $x_1 = 0$. Then, $x_2 \neq 0$ and there is exactly one other occurrence of 0 in x . Either x has the form 0t0 or the form 0t₁0t₂ for nonempty substrings t, t_1, t_2 . But if $x = 0t0$ then $t \in S_{12}$ has length 4 and necessarily contains Abelian squares and so x must as well. Therefore, $x = 0t_10t_2$ where $t_1, t_2 \in S_{12}$. Further, $|t_1| + |t_2| = 4$ and since both t_1 and t_2 are square-free, $|t_1|, |t_2| \leq 3$.

If $|t_1| = 1$ let $|t_2| = a \in \{1, 2\}$. Then $x = 0a0t_2$ is strongly square-free. It follows that $t_2 = bab \in S_{1,2}$, where $b \neq a$. We conclude that $x = 010212$ and $x = 020121$ are the only non-central strongly square-free strings in this case.

If $|t_1| = 2$ then $|t_2| = 2$. But if $t_1 = ab \in S_{1,2}$ then $t_2 = ab$ or $ba \in S_{1,2}$. In either case x is an Abelian square.

If $|t_1| = 3$ then $|t_2| = 1$ and $t_1 = aba$ or $bab \in S_{1,2}$. We see that x is either an Abelian square, or is central contrary to hypothesis.

We conclude that $x = 010212$ and $x = 020121$ are the only non-central strongly square-free strings beginning with 0.

Making the same arguments for x beginning with 1 we further obtain

101202 121020.

And if x begins with 2, the only possibilities are

202101 212010.

Therefore, there are only 6 strongly square-free strings of length 6 over $\{0, 1, 2\}$.

□

Lemma 4 *There are 6 non-central strongly square-free strings of length 7 over $\{0, 1, 2\}$.*

Proof. If x is any strongly square-free non-central string with $|x| = 7$ then x has one entry which appears exactly three times, since every entry must appear at least twice. Assume that $x_1 = 0$. Either 0 appears twice or three times in x . First suppose 0 appears three times. Then there are two cases: either x has the form $x = 0t_10t_20$ or $x = 0t_10t_20t_3$ where the strongly square-free substrings t_1, t_2, t_3 are in S_{12} . If $x = 0t_10t_20$, then $|t_1| + |t_2| = 4$ while $|t_1|, |t_2| \leq 3$. If $|t_1| = 1$ so $t_1 = a \in S_{12}$ then, as in the case of Lemma 3, we argue that $x = 0a0bab0$ which has an Abelian square suffix. If $|t_1| = 2$ then $t_1 = ab \in S_{12}$ and $x = 0ab0ba0$ which contains Abelian squares. The same is true in the last case: If $|t_1| = 3$ then $|t_2| = 1$ and all possibilities contain Abelian squares.

If $x = 0t_10t_20t_3$ then $|t_1| + |t_2| + |t_3| = 4$ while $1 \leq |t_1|, |t_2|, |t_3| \leq 3$. The only possibility is that exactly one of the substrings has length 2 and the others length 1. First suppose that $|t_1| = 2, |t_2| = 1, |t_3| = 1$. It follows that $x = 0ab0a0b$ or $x = 0ab0b0a$. But both of these contain Abelian squares. The second case is only somewhat different. If $|t_1| = 1, |t_2| = 2, |t_3| = 1$, then $t_2 = ab$ implies $t_1 = b$ since otherwise x would have a square prefix $0a0a$. Thus, either $x = 0b0ab0a$ or $x = 0b0ab0b$.

But in the first case, x contains the square $b0a$ as suffix and in the second case x would be central, contrary to hypothesis. The last case $|t_1| = 1, |t_2| = 1, |t_3| = 2$ is like the first since all possible strings contain Abelian squares.

Now suppose that 0 appears only twice in x . Then $x = 0t0$ or $x = 0t_10t_2$. The first case is impossible since t would belong to S_{12} while $|t| = 5$. If $x = 0t_10t_2$ then $|t_1| + |t_2| = 5$ while $0 \leq |t_1|, |t_2| \leq 3$. Consequently, there are only two cases to consider.

If $|t_1| = 3$ then $t_1 = aba$ and $t_1 = ab$ or ba for $a \neq b \in \{1, 2\}$. Thus, $x = 0aba0ab$ or $x = 0aba0ba$. In the later case x would have an Abelian square prefix. In the former case, we have only the two strings:

$$0121012 \quad 0212021$$

If $|t_1| = 2$ then $|t_2| = 3$ and it is easy to check that all possibilities contain Abelian squares.

We conclude that 0121012 and 0212021 are the only non-central strings starting with 0 of length 7 that are strongly square-free.

In the same way, if x begins with 1 we obtain strongly square-free strings 1020102 and 1202120. If x begins with 2 we further obtain strongly square-free strings 2010201 and 2101210. $\square$

Theorem 1 *There are 117 distinct strongly square-free strings on any alphabet of three letters.*

Proof. Each of the 3 sets,

$$S_{12}0S_{12} \quad S_{02}1S_{02} \quad S_{01}2S_{01}$$

contain 36 strongly square-free strings of length k , $3 \leq k \leq 7$ and intersect pair-wise in 4 strings. Since there are no strings in the intersection of all 3 sets, it follows that there are 96 distinct strings in the union of these sets. There do not exist non-central strings of length $k \leq 6$ and Lemmas 3 and 4 yield 12 non-central strings of lengths 6 and 7. Finally, since the alphabet provides 3 strongly square-free strings and there are 6 of length 2 we conclude that there are exactly 117 strongly square-free strings on three letters. $\square$

4 Generalizations

The case of just three letters is special because, as already remarked, it is the last case in which all strongly square-free strings are finite. Lemma 1 shows that there will exist finite strongly square-free strings over any finite alphabet. If empty strings are taken into account, then Lemma 1 shows that there are finite strongly square-free strings of every length $n \geq 1$ and gives a recursive way to construct many of them. In particular, it shows that there are at least 13,689 central strongly square-free finite strings over an alphabet of four letters. Obviously, central strings of all orders can be constructed, but the analysis for non-central strongly square-free strings is not yet clear.

Strongly Square-Free Finite Strings on $\{0, 1, 2\}$

	0	1	2
01	10	02	20 12 21
010	101	020	202 121 212
012	102	210	021 201 120
1012	0102	0201	
1021	0120	0210	
2021	2102	1201	
2012	2120	1210	
1202	0212	0121	
2101	2010	1020	
10121	01020	02010	
10212	01202	02101	
20121	21020	12010	
20212	21202	12101	
12021	02120	01210	
12012	02102	01201	
21021	20120	10201	
21012	20102	10210	
12102	02012	01020	
21201	20210	10120	
120121	120212	210121	210212
121021	121012	212012	212021
021020	021202	201020	201202
020120	020102	202102	202120
012010	012101	102010	102101
010210	010201	101201	101210
010212	020121	101202	121020
202101	212010		
1210121	1210212	2120121	2120212
0201020	0201202	2021020	2021202
0102010	0102101	1012010	1012101
0121012	0212021	1020102	1202120
2010201	2101210		

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