Directed Packings with Block Size 5 and Odd v

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Abstract

Let $DD(5,1;v)$ denote the packing number of a directed packing with block size 5 and index unity. It has been determined in [10] that $DD(5,1;v) = \lfloor \frac{v}{5} \lfloor \frac{2(v-1)}{4} \rfloor \rfloor$ where v is even. In this paper, the values of $DD(5,1;v)$ for all odd v are determined, with the possible exceptions of $v = 15,\ 19,\ 27$.

1 Introduction

Let v and k be positive integers. A transitively ordered k-tuple $(a_1, a_2, \ldots, a_k)$ is defined to be the set $\{(a_i, a_j) : 1 \leq i < j \leq k\}$ consisting of $k(k-1)/2$ ordered pairs. A directed packing with parameters v, k and $\lambda = 1$, denoted by $DP(k,1;v)$, is a pair $(X, \mathcal{A})$ where X is a v-set (of points) and $\mathcal{A}$ is a collection of transitively ordered k-tuples of X (called blocks) such that every ordered pair of distinct points of X occurs in at most one block of $\mathcal{A}$. If there is no other packing with more blocks, the packing is said to be maximum, and the number of blocks in a maximum packing is the packing number denoted by $DD(k,1;v)$.

In graph theoretic terms, a $DP(k,1;v)$ is equivalent to the decomposition of the complete symmetric directed graph K_v^* into transitive directed subgraphs on k vertices (tournaments of order k), in which some arcs of K_v^* are not used. The main problem asks for the maximum number of subgraphs i.e. $DD(k,1;v)$ in such a decomposition.

Using a simple counting argument similar to the Schönheim bound [9], we can show that the following inequality holds,

$$DD(k,1;v) \leq \left\lfloor \frac{v}{k} \left\lfloor \frac{2(v-1)}{k-1} \right\rfloor \right\rfloor \tag{1.1}$$

where $\lfloor x \rfloor$ is the floor of x. In what follows, the right hand side of (1.1) is denoted by $DU(k,1;v)$.

D. B. Skillicorn [11] studied in depth the function $DD(k,1;v)$ and determined completely values of $DD(3,1;v)$ and asymptotically for $DD(4,1;v)$. He also showed applications of this problem to coding theory and computer architecture reliability testing.

For some values of v the blocks of a maximum $DP(k,1;v)$ may contain every ordered pair of distinct points exactly once, in which case $DD(k,1;v) = \frac{2(v-1)v}{k(k-1)} = DU(k,1;v)$ and the packing is referred to as directed balanced incomplete block design ($DBIBD$) and denoted by $DB(k,1;v)$. D. J. Street and W. H. Wilson [12] proved that a $DB(5,1;v)$ exists when $v \equiv 1$ or $5 \pmod{10}$ with the exception of the non-existent $DB(5,1;15)$. This gives rise to the following

Theorem 1.1 Let $v \geq 5, v \neq 15$ and $v \equiv 1$ or $5 \pmod{10}$. Then $DD(5,1;v) = DU(5,1;v)$.

The values of $DD(5,1;v)$ for all even v were determined in [10] by N. Shalaby and J. Yin. In this paper we are interested in determining the values of $DD(5,1;v)$ for all odd v. The result is established, for the most part, by means of a result on directed balanced incomplete block designs, which is of interest in its own right.

2 Structure

In this section we discuss the structure of the maximum $DP(5,1;v)$ for the case of v is odd.

We first observe that if we ignore the order of the elements in the blocks, a $DP(5,1;v)$ becomes a standard $(v,5,2)$ packing. With this observation we have the following result from [13].

Lemma 2.1 If $v \equiv 7$ or $9 \pmod{10}$, then $DD(5,1;v) \leq DU(5,1;v) - 1$.

Let $(X,\mathcal{A})$ be a $DP(5,1;v)$. By $DR(X,\mathcal{A})$ we mean the directed graph spanned by the arcs that are not packed in $(X,\mathcal{A})$. It is clear that the number of arcs in $DR(X,\mathcal{A})$ is $v(v-1) - 10|\mathcal{A}|$. For any particular point x of X, there are exactly $2(v-1)$ ordered pairs containing x. When x appears in a block, it is included in the four ordered pairs contained in that block. The degree (including the indegree and outdegree) of x in $DR(X,\mathcal{A})$ therefore must be $2(v-1) - 4r_x$ where r_x is the number of blocks containing x.

Thus we have the following structure Lemma.

Lemma 2.2 Let $v \geq 5$ be an odd integer. Suppose that $(X, \mathcal{A})$ is a $DP(5,1;v)$ satisfying the following properties:

(1) $|\mathcal{A}| = DU(5,1;v) - 1$ when $v \equiv 7$ or $9 \pmod{10}$

(2) $|\mathcal{A}| = DU(5,1;v)$, otherwise.

Then the degree of each vertex of $DR(X, \mathcal{A})$ must be divisible by 4 and the number of arcs in $DR(X, \mathcal{A})$ is 0, 6 or 12 depending on whether $v \equiv \{1,5\}$, $\{3\}$ or $\{7,9\} \pmod{10}$, respectively.

We further have the following

Lemma 2.3 Let $v \equiv 3 \pmod{10}$. If $(X, \mathcal{A})$ is a $DP(5,1;v)$ with $DU(5,1;v)$ blocks, then the number of vertices in $DR(X, \mathcal{A})$ is 3.

Proof: From Lemma 2.2, we know that the number of vertices in $DR(X, \mathcal{A})$ must be at least 3. We claim that it must be at most 3. Assume that $DR(X, \mathcal{A})$ contains 4 vertices. Since $v \equiv 3 \pmod{10}$, for any $x \in X$, the number of blocks containing x is $r_x \leq \left\lfloor \frac{2(v-1)}{4} \right\rfloor = \frac{v-1}{2}$. Therefore $|\mathcal{A}| = \sum_{x \in X} \frac{r_x}{5} \leq \frac{[(v-4)\frac{v-1}{2} + 4(\frac{v-1}{2} - 1)]}{5} = \frac{(v(v-1)-8)}{10} < DU(5,1;v)$ a contradiction.

The only directed graph with 6 arcs, 3 vertices and each vertex of degree divisible by 4 is the directed complete graph K_3^*. The situation is more complicated when $v \equiv 7$ or 9 (mod 10) since there are many directed graphs with 12 arcs satisfying the degree constraint. In this case, we shall use the following result.

Lemma 2.4 If $v = 7$ or 9, then $DD(5,1;v) = DU(5,1;v) - 1$.

Proof: In view of Lemma 2.1, we need only construct $DP(5,1;v)$ with $DU(5,1;v) - 1$ blocks. For the stated values of v, we define the point set to be $X = \{1,2,3,\ldots,v\}$. Then the required packings are obtained by taking the following blocks:

$v = 7 : (1,2,3,4,5)(6,7,5,4,3)(3,2,1,7,6)$

$v = 9 : (1,2,4,6,7)(2,8,9,1,5)(3,5,7,6,1)(4,1,3,9,8)(6,5,9,2,3)(7,8,3,4,2)$.

Foregoing can be summarized in the following theorem.

Theorem 2.5 Let $v \geq 5$ be an odd integer. Then

(1) a $DB(5,1;v)$ with a hole of size 3 exists if and only if $DD(5,1;v) = DU(5,1;v)$ whenever $v \equiv 3 \pmod{10}$

(2) $DD(5,1;v) = DU(5,1;v) - 1$ if $v \equiv 7, 9 \pmod{10}$ and a $DB(5,1;v)$ with a hole of size 7 or 9 exists.

3 Constructions for Incomplete Directed BIBDs

In order to describe our constructions we require a number of types of other combinatorial designs. For the definitions of group divisible designs (GDDs), see [4,7]. We use notation K-GDD to indicate a GDD with index unity and block sizes from K. When $K = \{k\}$, we omit the braces. The type of GDD is a listing of its group sizes and we use the so-called "exponential" notation: a type $1^i 2^j 3^k \ldots$ denotes i occurrences of groups of size 1, j occurrences of groups of size 2, and so on. In the literature a k-GDD of type m^k is often called a transversal design (TD) and is denoted by $TD(k, m)$. And a k-GDD of type 1^v is often called a balanced incomplete block design (BIBD) and denoted by $B(k, 1; v)$. Furthermore, a k-GDD of type $1^u w^1$ is referred to as an incomplete BIBD. The group of size w is the hole. We write it by $IB(k, 1; u + w, w)$.

We now define a directed group divisible design (DGDD). A k-DGDD is a triple $(X, \mathcal{G}, \mathcal{A})$, where

1) X is a finite set (of points),

2) $\mathcal{G}$ is a collection of subsets of X (called groups) which partition X,

3) $\mathcal{A}$ is a collection of transitively ordered k-subsets of X (called blocks),

4) no block meets a group in more than one point, and

5) each ordered pair (x, y) of distinct points not contained in the same group occurs in exactly one of the blocks.

Similar to GDDs, the type of a DGDD is a listing of its group sizes and is denoted by the "exponential" notation. A k-DGDD of type $1^u w^1$ is defined to be an incomplete DBIBD and written as $IDB(k, 1; u + w, w)$. The group of size w is the hole. It is clear that an $IDB(k, 1; u + w, w)$ with $w = 1$ is essentially a $DB(k, 1; u + 1)$.

In the sequel we shall use the following existence theorems for DGDDs whose proofs can be found in [10].

Lemma 3.1 Let $n \geq 5$ be an integer satisfying $n \neq 6, 10, 14, 18, 22, 34$ or 42. Then there exists a 5-DGDD of type $(2n)^5(2s)^1$ where s is any integer satisfying $0 \leq s \leq n$.

The following construction is an extension of construction 4.5 in [13] for incomplete DBIBD's.

Lemma 3.2 Let u, e and w be nonnegative integers. Suppose that there exist an $IB(k, 1; u + 2e + w, 2e + w)$ and an $IB(k, 1; u + w, w)$. Then there exists an $IDB(k, 1; u + e + w, e + w)$.

Proof: The case of $e = 0$ is trivial, thus we assume that $e \geq 1$. Let $\mathcal{B}_1$ be the collection of $IB(k, 1; u + 2e + w, 2e + w)$ defined on $I(u + 2e + w) = \{1, 2, \ldots, u + 2e + w\}$ with the hole $\{u + 1, u + 2, \ldots, u + 2e + w\}$. We use $\mathcal{B}_1$ to create a collection $\mathcal{A}_1$ of transitively ordered k-tuples by the following way.

First, we order each block of $\mathcal{B}_1$ which does not contain the symbol $u+e+w+i (1 \le i \le e)$ in a strictly increasing order.

Secondly, we order each block which contains the symbol $u+e+w+i (1 \le i \le e)$ so that the symbol $u+e+w+i$ lies on the leading place and the other symbols are in a strictly increasing order.

Finally, we replace the symbol $u+e+w+i$ by the symbol $u+i$, for $1 \le i \le e$, wherever it occurs.

Then we construct by the hypothesis an $IB(k, 1; u+w, w)$ on the set $[I(U)] \cup \{u+e+j : 1 \le j \le w\}$. Write $\mathcal{B}_2$ for its block collection. We now order each block of $\mathcal{B}_2$ in a strictly decreasing order to create another collection $\mathcal{A}_2$ of transitively ordered k-tuples. Thus an $IDB(k, 1; u+e+w, e+w)$ defined on $I(u+e+w)$ with a hole $\{u+1, u+2, \ldots, u+e+w\}$ is obtained by taking the collection of blocks $\mathcal{A}_1 \cup \mathcal{A}_2$.

The following result is an extension of the well-known technique of "Filling Holes", of incomplete DBIBDs. Please refer to [10] pp. 136-137 and [13].

Lemma 3.3 Suppose that there exist a k-DGDD of type $\{t_1, t_2, \ldots, t_n\}$ and an $IDB(k, 1; t_i + w, w)$ for $1 \le i \le n-1$. Then there exists an $IDB(k, 1; t+w, t_n+w)$ where $t = \sum t_i$. Furthermore, an $IDB(k, 1; t+w, e)$ exists if an $IDB(k, 1; t_n+w, e)$ exists.

Combining Lemma 3.3 with the results of Lemma 3.1 gives rise to the following.

Lemma 3.4. Let $t \ge 5$ be an integer and $t \ne 6, 10, 14, 18, 22$ or 42. Then there exists an $IDB(5, 1; 10t+2s+w, 2s+w)$ if an $IDB(5, 1; 2t+w, w)$ exists. Moreover, an $IDB(5, 1; 10t+2s+w, e)$ exists if an $IDB(5, 1; 2s+w, e)$ exists, where s is any integer satisfying $0 \le s \le t$.

We are now in the position to present our results for incomplete DBIBDs. For ease of notation we define

$$IDB(w) = \{v : \text{ there is an } IDB(5, 1; v, w)\}.$$

Lemma 3.5 (1) $\{v : v \equiv 3 \pmod{20} \text{ and } v \ge 23\} \subset IDB(3)$,

(2) $\{v : v \equiv 7 \pmod{20} \text{ and } v \ge 47\} \subset IDB(7)$, (3) $\{v : v \equiv 9 \pmod{20} \text{ and } v \ge 49\} \subset IDB(9)$.

Proof: Note that a $B(5, 1; v)$ is equivalent to an $IB(5, 1; v, w)$ where $w = 1$ or 5. It is known [7] that a $B(5, 1; v)$ exists when $v \equiv 1$ or $5 \pmod{20}$. It is also known [6] that an $IB(5, 1; v, 13)$ exists when $v \equiv 13 \pmod{20}$ and $v \ge 53$. Thus suitable choices of u, e and w in Lemma 3.2 can be made to obtain the required result.

Lemma 3.6 (i) If $v \equiv 17 \pmod{20}$ and $v \ge 37$, then $v \in IDB(9)$.

(ii) $DD(5, 1; 17) = DU(5, 1; 17) - 1$.

Proof: (i) It is known [6] that an $IB(5,1;v,9)$ exists if $v \equiv 17 \pmod{20}$ and $v \geq 37$. For each stated value of v, we start with an $IB(5,1;v,9)$ and write every block twice — once in some order and the other in the reverse order. The result is an $IDB(5,1;v,9)$ and the proof is complete.

For (ii) let $X = \{1,2,\ldots,17\}$, then the blocks are

$(1,2,6,13,12)$ $(15,3,12,16,7)$ $(2,1,11,16,14)$ $(12,17,3,13,11)$ $(11,6,1,3,15)$

$(4,6,7,9,11)$ $(3,1,9,10,17)$ $(4,10,16,13,15)$ $(15,17,1,4,5)$ $(13,6,5,16,17)$

$(14,7,12,4,1)$ $(11,5,9,7,13)$ $(16,8,9,5,1)$ $(14,5,15,10,11)$ $(13,10,1,7,8)$

$(7,14,17,16,6)$ $(9,16,2,3,4)$ $(12,9,15,8,6)$ $(7,10,3,2,5)$ $(8,16,11,12,10)$

$(11,4,2,8,17)$ $(17,10,9,12,14)$ $(17,8,2,7,15)$ $(15,14,13,2,9)$ $(3,5,6,14,8)$

$(8,13,4,14,3)$

Lemma 3.7 $\{29,39,79\} \subset IDB(7)$ and $99 \subset IDB(9)$.

Proof: For $v = 29$ let $X = (Z_3 \cup \{\infty\}) \times Z_7 \cup \{\infty\}$where the hole is $\{\infty\} \times Z_7$. Then the blocks are:

$((0,0)(1,0)(2,0)(\infty,0),\infty) \bmod (-,7)$

$(\infty,(\infty,0)(2,0)(1,0)(0,0)) \bmod (-,7)$

$((1,-2\cdot 2^i)(0,2^i)(\infty,0)(0,-2^i)(1,2\cdot 2^i)) \bmod (3,7)$, $i = 0,1,2$.

For $v = 39$ let $X = Z_{32} \cup H_7$ where $H_7 = \{h_0,\ldots,h_6\}$ is the hole. Then the blocks are

$(13,10,h_i,25,0)$ translated by G_i, $i = 0,1$.

$(2,11,h_{i+2},15,0)$ translated by G_i, $i = 0,1$.

$(6,29,h_{i+4},0,5)$ translated by G_i, $i = 0,1$.

$(16,0,h_6,2,18)$ translated by $\{0,1,\ldots,15\}$

$(4,31,0,10,24) \bmod 32$,

where G_0 is the subgroup of even integers in Z_{32} and G_1 is the coset of odd integers in Z_{32}.

For $v = 79$ let $X = Z_{72} \cup H_7$ where $H_7 = \{h_0,\ldots,h_6\}$ is the hole. Then the blocks are

$(42,11,h_i,49,0)$ translated by G_i, $i = 0,1$.

$(0, 25, h_{i+2}, 42, 9)$ translated by G_i, $i = 0, 1$.

$(7, 22, h_{i+4}, 0, 33)$ translated by G_i, $i = 0, 1$.

$(50, 14, h_6, 0, 36)$ translated by $\{0, 1, \ldots, 35\}$

$(3, 5, 1, 9, 0)$ (mod 72), $(38, 0, 13, 18, 3)$ mod (72)

$(6, 26, 0, 27, 55)$ (mod 72), $(8, 18, 0, 53, 32)$ mod (72)

$(53, 24, 0, 40, 12)$ (mod 72)

where G_0 is the subgroup of even integers in Z_{72} and G_1 is the coset of odd integers in Z_{72}.

For $v = 99$, take an $RMGD[5, 1, 5, 45]$ (see [1] for the definition and existence of resolvable modified group divisible design RMGD) and inflate it by a factor of two. To each of the three parallel classes of quintuples add two points and replace their blocks by the blocks of a 5-DGDD of type 2^6 and on the remaining blocks construct a 5-DGDD of type 2^5. The two input designs exist. See for example [10], [12]. To the groups add a new point and replace their blocks by the blocks of a $DB(5, 1; 11)$. Finally, to the parallel class of size 9 add two points and replace their blocks by the blocks of a 5-DGDD of type 2^{10}. Such designs can be constructed as follows:

Let $X = Z_2 \times Z_9 \cup \{\infty_1, \infty_2\}$, the blocks are

$((0, 0)(0, 4)(1, 2)(0, 1)(0, 3))$ mod$(-, 9)$

$((1, 3)(0, 0)(1, 5)(1, 6)(1, 1))$ mod$(-, 9)$

$((1, 6)(0, 2)\infty_1(1, 5)(0, 0))$ mod$(-, 9)$

$((1, 2)(0, 4)\infty_2(0, 0)(1, 8))$ mod$(-, 9)$.

Lemma 3.8 (i) $DD(5, 1; 13) = DU(5, 1; 13) - 1$

(ii) $\{33, 73, 93, 113, 213, 313\} \subset IDB(3)$

Proof: (i) In [1] Assaf showed that a (5,2,13) packing design has $DU(5, 1; 13) - 1 = 14$ blocks. Hence $DD(5, 1; 13) \leq DU(5, 1; 13) - 1$. To show equality we construct a $DP(5, 1; 13)$ with 14 blocks.

Let $X = \{1, 2, \ldots, 13\}$ then the blocks are:

$(1, 2, 3, 5, 11)(12, 8, 10, 2, 1)(3, 1, 7, 8, 13)$ $(13, 5, 1, 10, 6)(7, 1, 9, 11, 12,)(7, 3, 2, 9, 10)$

$(6, 5, 9, 2, 8)$ $(11, 2, 6, 12, 13)(8, 3, 4, 12, 6)$ $(6, 11, 4, 10, 3)(9, 5, 12, 13, 3)(4, 8, 5, 7, 11)$

$(10, 12, 7, 5, 4)(10, 13, 11, 8, 9)$

For $v = 33$, let $X = Z_3 \times Z_{10} \cup \{\infty_i\}_{i=1}^{3}$.

The blocks are:

$((1,0)(1,5)(0,0)(0,5),\infty_1) + (-,i) i \in Z_5$

$(\infty_2,(0,5)(0,0)(2,0)(2,5)) + (-,i) i \in Z_5$

$((2,5)(2,0)(1,5)(1,0),\infty_3) + (-,i) i \in Z_5$

$((0,0)(1,0)(1,3)(0,2)(0,1)) \bmod (-,10)$ $((0,6)(2,0)(0,2)(0,0)(1,1)) \bmod (-,10)$

$((2,4)(1,6)(2,2)(0,0)(0,3)) \bmod (-,10)$ $((2,7)(2,6)(1,4)(0,0)(2,9)) \bmod (-,10)$

$((2,5)(0,0)(1,8)(2,6)(2,2)) \bmod (-,10)$ $((1,0)(1,4)(1,2)(1,1)(2,3)) \bmod (-,10)$

$((2,3)\infty_1(1,7)(0,0)(2,7)) \bmod (-,10)$ $((0,3)\infty_3(0,0)(2,1)(1,7)) \bmod (-,10)$

$((0,0)(1,6)(2,3),\infty_2(1,2)) \bmod (-,10)$.

For $v = 73$ let $X = Z_2 \times Z_{35} \cup \{\infty_i\}_{i=1}^{3}$. The blocks are:

$((0,0)(0,1)(0,3)(0,8)(0,20)) \bmod (-,35)$ $((1,0)(0,0)(0,25)(0,4)(0,13)) \bmod (-,35)$

$((1,2)(1,4)(1,9)(1,1)(0,0)) \bmod (-,35)$ $((1,20)(0,17)(0,0)(0,6)(0,27)) \bmod (-,35)$

$((1,23)(1,27)(1,17)(1,6)(0,0)) \bmod (-,35)$ $((1,21)(0,5)(0,0)(0,16)(1,34)) \bmod (-,35)$

$((0,4)(1,11)(1,19)(1,30)(0,0)) \bmod (-,35)$ $((1,13)(1,25)(1,34)(0,1)(0,0)) \bmod (-,35)$

$((0,2)(1,18)(0,0)(1,0)(1,3)) \bmod (-,35)$ $((0,3)(1,15)(0,0)(1,13)(1,8)) \bmod (-,35)$

$((0,6)(1,8)(0,0)(1,30)(1,31)) \bmod (-,35)$ $((0,7)(0,0)(1,11)(1,21)(1,27)) \bmod (-,35)$

$((0,9)(1,32)\infty_1(0,0)(1,28)) \bmod (-,35)$ $((1,7)(0,13)\infty_2(0,0)(1,22)) \bmod (-,35)$

$((0,0)(1,6)\infty_3(0,15)(1,32) \bmod (-,35)$.

For $v = 93$ let $X = (Z_3 \cup \{\infty\}) \times Z_{23} \cup \{\infty\}$, where the hole is $\{\infty\} \times Z_{23}$.

The blocks are:

$((0,0)(1,0)(2,0)(\infty,0),\infty) \bmod (-,23)$

$(\infty(\infty,0)(2,0)(1,0)(0,0)) \bmod (-,23)$

$((1,-4\cdot 2^i)(0,2^i)(\infty,0)(0,-2^i)(1,4\cdot 2^i)) \bmod (3,23)$.

For $v = 113$, we proceed as follows. In a $TD(6, 11)$, we delete all but one point from the last group to obtain a $\{5, 6\}$-GDD of type $11^5 1^1$. Write $\mathcal{A}$ for its block collection and let $G_1, G_2, \ldots, G_6$ be its groups, where $|G_i| = 11$, for $i = 1, 2, \ldots, 5$ and $|G_6| = 1$. In the GDD, we take a particular block B of size 5. It is clear that B is disjoint from G_6 and intersects any other group of the GDD in exactly one point. We then assume $B \cap G_i = \{x_i\}$ for $1 \leq i \leq 5$. Now let $X = \cup_{i=1}^{6} G_i$ and $Y = X \times I(2) \cup \{\infty\}$. Now for each block A other than B we construct a 5-DGDD of type 2^5 or 2^6 (depending upon $|A| = 5$ or 6) defined on the set $A \times I(2)$ taking $\{a\} \times I(2), a \in A$, as groups. Denote its block collection by $\mathcal{B}_A$. Let $\mathcal{A}_i, i = 1, 2, \ldots, 5$, denote the collection of blocks of a copy of an $IDB(5, 1; 23, 3)$ (see Lemma 3.5) formed on the set $G_i \times I(2) \cup \{\infty\}$ in such a fashion that the hole is $\{x_i\} \times I(2) \cup \{\infty\}$.

Finally we make use of Theorem 1.1 to construct a $DB(5, 1; 11)$ on the set $B \times I(2) \cup \{\infty\}$ and denote its block collection by $\mathcal{A}_\infty$. Then the blocks of $(U\mathcal{B}_{A_{A \in \mathcal{A} \setminus \{B\}}}) \cup (\cup_{i=1}^{5} \mathcal{A}_i) \cup \mathcal{A}_\infty$ form the block collection of an $IDB(5, 1; 113, 3)$ on the set Y whose hole is $G_6 \times I(2) \cup \{\infty\}$.

For the case $v = 213$ or 313, the proof is similar. In this case, we start with a $TD(6, 21)$ or a $TD(6, 31)$ instead of a $TD(6, 11)$.

It is worth noting that the 5-DGDDs used above come from [7, Lemma 5] and all TDs used above exist (see, for example, [7]).

Lemma 3.9 (1) $\{v : v \equiv 13 \pmod{20} \text{ and } v \geq 33\} \subset IDB(3)$;

(2) $\{v : v \equiv 19 \pmod{20} \text{ and } v \geq 39\} \subset IDB(7) \cup IDB(9)$.

Proof: From our previous Lemmas, we know that the conclusion holds for $v \in \{33, 73, 93, 39, 79, 99\} \cup \{113, 213, 313\}$. Applying Lemma 3.4 with the parameters shown in Table 1 and the previous established results we obtain that the conclusion holds for $v \leq 339$. We now apply Lemma 3.4 for each $t \geq 35$ and $t = 15$ or $17 \pmod{20}$, $w = 1$ and $s \in \{1, 11, 21, 31\}$ for the case $v \equiv 13 \pmod{20}$ and $s \in \{4, 14, 24, 34\}$ for the case $v \equiv 19 \pmod{20}$. This guarantees the conclusion holds for $v \geq 353$ because of the Lemma 3.5 and the existence of a $DB(5, 1; v)$ mentioned in Section 1.

Table 1

$10t + 2s + w$	t	s	w	$10t + 2s + w$	t	s	w
53/59	5	1/4	1	233/239	21	8/11	7
119	11	1	7	253/259	21	18/21	7
133/139	11	8/11	7	273/279	27	1/4	1
153/159	15	1/4	1	293/299	27	11/14	1
173/179	15	11/14	1	319	31	1	7
193/199	17	11/14	1	333/339	31	8/11	7
219	21	1	7				

Summarizing the above results, we have proved

Theorem 3.10 (1) $v \geq 23$ and $v \equiv 3 \pmod{10}$, then $v \in IDB(3)$.

(2) If $v \geq 29$ and $v \equiv 7$ or $9 \pmod{10}$, then $v \in IDB(7) \cup IDB(9)$.

Combining the results of Theorems 1.1, 2.5 and 3.10, we have

Theorem 3.11 Let $v \geq 5$ be an odd integer and $v \notin \{15, 19, 27\}$. Then $DD(5,1;v) = \begin{cases} DU(5,1;v) - 1, & \text{if } v \equiv 7, 9 (\text{mod } 11) \text{ or } v = 13 \\ DU(5,1;v), & \text{otherwise .} \end{cases}$

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