Steiner Minimal Trees on the Union of Two Orthogonal Rectangles

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Abstract

Suppose R is a union of two subsets R_1 and R_2 whose Steiner minimal trees $SMT(R_1)$ and $SMT(R_2)$ are known. The decomposition question is when the Steiner minimal tree $SMT(R)$ for R is just the union of two Steiner minimal trees on R_1 and R_2 respectively. In this paper a special case is studied, that is, $R_1 = bcda$, $R_2 = defg$ are two non-overlapping rectangles with a common vertex d so that a,d,e lie on one line. We conclude that $SMT(R)$ has only two possible structures. We also give two sufficient conditions for the required decomposition $SMT(R_1 \cup R_2) = SMT(R_1) \cup SMT(R_2)$, and prove that under suitable assumptions of randomness, the probability of such a decomposition is 0.9679.

1. Introduction

The Steiner problem for a given set R of points (called *regular points*) in the Euclidean plane is to construct a shortest network interconnecting these given points, with some additional points (called *Steiner points*) [2]. The shortest network is a tree, called the *Steiner minimal tree* for R, and denoted by $SMT(R)$. If the degree of every regular point is one, then the tree is called *full*. All angles in Steiner minimal trees are no less than 120°. This is called the *angle condition* of Steiner minimal trees.

As in other fields of mathematics, the following *decomposition question* also can be raised in this shortest network problem: If R is a union of several simple subsets R_i, $i = 1, 2, ..., k$, whose Steiner minimal trees are known, then when do we have

$$SMT(R) = SMT(\bigcup_{i=1}^{k} R_i) = \bigcup_{i=1}^{k} SMT(R_i)?$$

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Clearly, we should give some restraints on the subsets. In this paper we study a special case: $i = 2$ and R_i are rectangles. More specifically, suppose $R = R_1 \cup R_2$ where $R_1 = bcda$, $R_2 = defg$ are two rectangles with a common vertex d so that a, d, e lie on one line. (For convenience, we assume that the line is horizontal.) Then R is called a union of two *orthogonal rectangles* (Fig. 1).

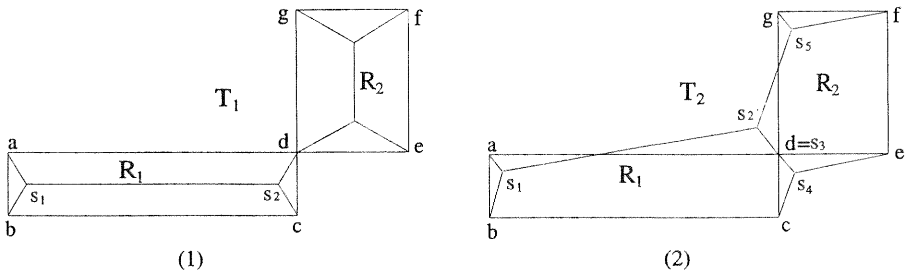

Figure 1

In this paper we prove that, up to symmetry, $SMT(R)$ has only two possible structures. Then we give two sufficient conditions for the required decomposition $SMT(R_1 \cup R_2) = SMT(R_1) \cup SMT(R_2)$, and finally, prove that under suitable assumptions of randomness, the probability of such a decomposition is 0.9679.

2. Steiner minimal trees for R

By the topology we mean the structure of the network. It has been proved that we need only consider full Steiner topologies in order to determine Steiner minimal trees [3]. Usually the vertices of an angle or a polygon are written in counterclockwise order. Following Cockayne [1], we denote by (ag) the third vertex of the equilateral triangle $\triangle(ag)ag$. Hence, the point (ag) is on the left side looking from a to g. Note that by Melzak's construction [4], the Simpson line of a full Steiner tree can also be expressed by this notation.

A path $ps_1s_2...s_mq$ is called a *left-* (or *right-*)*turn* path (*starting with edge* ps_1) if it always turns left (or right) at every vertex s_i, $1 \leq i \leq m$, on the path. It is called a *Steiner path* if all s_i are Steiner points. Suppose $ps_1s_2...s_mq$ is a convex polygon and point a is outside it and on the same side of pq as all s_i. Then we call the path $ps_1s_2...s_mq$ *convex to a*. The following general lemma is easily seen.

Lemma 1. Suppose two lines l_1 and l_2 meet at a regular point a at a right angle.

(1) Then no one edge of the Steiner minimal tree can intersect both l_1 and l_2.

(2) If there is a Steiner path which is convex to a and intersects l_1 and l_2 at p and r respectively, then there is one and only one Steiner point s between p and r.

Moreover, the angle between l_1 and sp, as well as the angle between l_2 and sr, are both less than $30°$.

The length of an edge or a tree is denoted by $|\cdots|$.

Corollary 1. In the Steiner minimal tree T of R, the degree of d is no more than two, and the degree of all other regular points is one.

Proof. If there are two edges at b (or f), the angle between them is less than $90°$. If there are two edges at a (or c, e, g), then the angle between them is less than $120°$ by Lemma 1. In both cases the angle condition of Steiner minimal trees is contradicted. Suppose the degree of d is three. By the angle condition we may assume without loss of generality that one Steiner point of d lies in $\angle adg$ and the other two Steiner points lie in R_1, R_2 respectively. Then the tree T must be $SMT(adg) \cup SMT(bcd) \cup SMT(def)$. Since the three angles at d are all equal to $120°$, it is easy to see that $|ad| = |de|, |cd| = |dg|$. It follows that $|T| > |T_1|$, T_1 as in Figure 1. T is not minimal. ∎

Among the different trees, we consider in particular the Steiner trees T_1, T_2, T_2^* given by (see Fig. 1)

$$T_1 = SMT(R_1) \bigcup SMT(R_2),$$

$$T_2 = (ba)((gf)d) \bigcup (cd)e,$$

$$T_2^* = (da)g \bigcup (cb)((df)e).$$

Clearly, the topology of T_2^* is symmetric to the topology of T_2. Define

$$f(x,y) = \sqrt{x^2 + xy\sqrt{3} + y^2}.$$

Lemma 2. $f(x,y) > x\sqrt{3}/2 + y$, for $x > 0, y > 0$.

Proof. It can be verified directly.

Theorem 1. Up to symmetry, the Steiner minimal tree for R, is either T_1 or T_2.

Proof. Suppose T is a Steiner minimal tree for R. Let the path from b to a be $bs'_1 \cdots s'_{k_1}a$ with k_1 Steiner points, the path from b to c be $bs_1 \cdots s_{k_2}c$ with k_2 Steiner points. By Corollary 1, $s_1 = s'_1, k_1 \geq 1, k_2 \geq 1$, and at most one of k_1, k_2 equals one. So $k_1 + k_2 \geq 3$. Since there are 5 Steiner points in a full Steiner tree for R and the Steiner point adjacent to f must lie in R_2, $k_1 + k_2 \leq 5$. Without loss of generality assume $k_1 \leq k_2$. There are just 5 cases to consider.

(1) $k_1 = 1, k_2 = 2$. By Lemma 1 the third edge of s_2 can neither intersect cd and ad, nor end in R_1. Hence, s_2 joins d, and consequently, $T = T_1$.

(2) $k_1 = 1, k_2 = 3$. s_3 lies in $\triangle edc$ and the third edge of s_3 meets de at a point, say p. Again by Lemma 1 the third edge of s_2 can neither intersect ad, nor end in R_1. It cannot end in $\triangle edc$, otherwise one of the right-turn paths starting with the third edge of s_2 and s_3 ends nowhere. Hence, s_2 joins d or $s_2 = d$. In the former case let q be the intersection of s_2s_3 and cd, and let q' be the point on dc such that $|dq| = |q'c|$. Then

$$|((ba)d)(pc)| = |(ba)(dq)| + |SMT(pqc)| > |(ba)(q'c)| + |SMT(pdq')|.$$

Hence, T is not minimal. In the latter case, $|ad| \geq |dp|$ since $\angle s_3ds_1 \geq 120°$. By Lemma 2 using Melzak's construction

$$\begin{aligned} |SMT(abd)| + |SMT(dcp)| &= f(|ab|, |ad|) + f(|dc|, |dp|) \\ &\geq \frac{\sqrt{3}}{2}|ab| + |ad| + \frac{\sqrt{3}}{2}|dc| + |dp| \\ &= |SMT(abcd)| + |dp|. \end{aligned}$$

However, if a tree contains $SMT(abcd) \cup dp$ as its part, then either the degree of d is three or the degree of e is two. Hence, Corollary 1 is contradicted either for d or for e. This means that $SMT(abd) \cup SMT(dcp)$ is not a minimal tree spanning $\{a, b, c, d, p\}$, and hence, T is not minimal either.

(3) $k_1 = 2, k_2 = 2$. Since the third edge of s_2' and s_2 are parallel, one has to meet ad and another has to meet dc. Hence, one of them contradicts Lemma 1.

(4) $k_1 = 1, k_2 = 4$. By the angle consideration it is easy to see that one of $s_1, \ldots, s_4$, and in fact s_3, should collapse into d. It follows that s_2 lies in $\triangle adg$ and s_4 lies in $\triangle edc$. There are two possibilities. If s_2 joins g and s_4 joins another Steiner point s_5 which is adjacent to both ef, then it is easily seen that the tree $T = SMT(abdg) \cup SMT((dcef)$ is longer than T_1. If s_4 joins e and s_2 joins another Steiner point which is adjacent to both g and f, then $T = T_2$.

(5) $k_1 = 2, k_2 = 3$. If no Steiner point of s_1, s_2, s_3 collapses into d then $\angle s_2'ab + \angle bcs_3 = 270°$ by considering the sum of the interior angles of $abcs_3s_2s_1s_2'$.

Lemma 1 is then contradicted. However, if $s_2 = d$, then the subtree spanning $abdp = (ap)(db)$ is longer than $(ba)(pd)$ where p is the intersection of dg with the third edge of s'_2. So T is not minimal. ∎

3. Two sufficient conditions for $SMT(R) = SMT(R_1) \bigcup SMT(R_2)$

Let the widths and heights of R_i be w_i and h_i $(i = 1, 2)$ respectively. Because the Steiner minimal tree is only concerned with in the relative position of two orthogonal rectangles, we may assume without loss of generality that w_1 is the largest of w_1, h_1, w_2, h_2. Let s_2 be the Steiner point in T_2 which lies in $\triangle adg$.

Lemma 3. T_2 exists, i.e., the Steiner point s_2 does not collapse into d, if and only if $h_1/w_1 < h_2/w_2$. By symmetry, T_2^* exists if and only if $h_1/w_1 > h_2/w_2$.

Proof. Let $\phi_1 = \angle(ba)da$, $\phi_2 = \angle gd(gf)$. We need to prove that $\phi_1 + \phi_2 < 30°$ if and only if $h_1/w_1 < h_2/w_2$. Let $\gamma_1 = \angle bda, \gamma_2 = \angle gdf$. It is easily shown that

$$\cot \phi_1 = 2 \cot \gamma_1 + \sqrt{3}, \quad \cot \phi_2 = 2 \cot \gamma_2 + \sqrt{3}.$$

Then $\phi_1 + \phi_2 < 30°$ if and only if

$$\begin{aligned}\cot(\phi_1 + \phi_2) &= \frac{\cot \phi_1 \cot \phi_2 - 1}{\cot \phi_1 + \cot \phi_2} \\ &= \frac{(2 \cot \gamma_1 + \sqrt{3})(2 \cot \gamma_2 + \sqrt{3}) - 1}{2 \cot \gamma_1 + 2 \cot \gamma_2 \sqrt{3}} \\ &> \sqrt{3}.\end{aligned}$$

This inequality is equivalent to $\cot \gamma_1 \cot \gamma_2 > 1$, i.e., $h_1/w_1 < h_2/w_2$. ∎

Since only one of T_2 and T_2^* can exist by Lemma 3, by symmetry, we assume that T_2 exists, (i.e., $h_1/w_1 < h_2/w_2$) from now on.

Lemma 4. $f(x, y) + x > \sqrt{3}x + y$, for $x > 0, y > 0$.

Proof. It can be verified directly by the definition of $f(x, y)$.

$$|T_2| = f(h_1 + h_2, w_1 + w_2) + f(h_1, w_2).$$

Theorem 2. If $h_2 \le w_2$, then $|T_1| < |T_2|$.

Proof. First we assume $h_2 = w_2$. By Lemma 4 we have

$$\begin{aligned}|T_2| &= f(h_1 + h_2, w_1 + w_2) + f(h_1, w_2)\\ &> f(h_1 + h_2, w_1 + w_2) + (h_1 + w_2)\\ &> \sqrt{3}(h_1 + h_2) + (w_1 + w_2) = |SMT(R_1)| + |SMT(R_2)| = |T_1|.\end{aligned}$$

Now suppose $h_2 < w_2$. Let s_5, s_4 be the Steiner points incident to f, e respectively (Fig. 1(2)). We shrink de and gf till $w_2 = h_2$. Note that both $\angle s_5 fg$ and $\angle s_4 ed$ are less than 30° by Lemma 1(2).

$$\begin{aligned}\frac{\partial |T_2|}{\partial w_2} &= -(\cos \angle s_5 fg + \cos \angle s_4 ed)\\ &< -1 = \frac{\partial |T_1|}{\partial w_2}.\end{aligned}$$

Hence, by the variational argument [5] we have $|T_2| > |T_1|$. ∎

Lemma 5.

$$f(x, y) \geq \left(\frac{x + y}{2}\right)\sqrt{2 + \sqrt{3}}, \text{ for } x > 0, y > 0.$$

The equality holds if and only if $x = y$.

Proof. Put $x' = y' = (x + y)/2$. Then $x'y' = (x + y)^2/4 \geq xy$, and equality holds if and only if $x = y$. So,

$$\begin{aligned}f(x, y) &= \sqrt{x^2 + xy\sqrt{3} + y^2}\\ &= \sqrt{(x + y)^2 - (2 - \sqrt{3})xy}\\ &\geq \sqrt{(x' + y')^2 - (2 - \sqrt{3})x'y'}\\ &= \sqrt{x'^2 + x'y'\sqrt{3} + y'^2}\\ &= x'\sqrt{2 + \sqrt{3}} = \left(\frac{x + y}{2}\right)\sqrt{2 + \sqrt{3}}.\end{aligned}$$

∎

Theorem 3. Suppose $h_2 > w_2$. Then $|T_1| < |T_2|$ if

$$\frac{h_1 + w_2}{w_1 + h_2} > \frac{2 - \sqrt{2 + \sqrt{3}}}{2(\sqrt{2 + \sqrt{3}} - \sqrt{3})} \ (\approx 0.17). \quad (1)$$

Proof. Since $h_2 > w_2$, $|T_1| = 1 + h_2 + h_1\sqrt{3} + w_2\sqrt{3}$. Then

$$\begin{aligned}
|T_2| &= f(h_1 + h_2, w_1 + w_2) + f(h_1, w_2) \\
&\geq \left(\frac{h_1 + h_2 + w_1 + w_2}{2}\right)\sqrt{2+\sqrt{3}} + \left(\frac{h_1 + w_2}{2}\right)\sqrt{2+\sqrt{3}} \\
&= (w_1 + h_2)\left(\frac{\sqrt{2+\sqrt{3}}}{2}\right) + (h_1 + w_2)\sqrt{2+\sqrt{3}} \\
&= (w_1 + h_2) + (h_1 + w_2)\sqrt{3} \\
&\qquad + (h_1 + w_2)\left(\sqrt{2+\sqrt{3}} - \sqrt{3}\right) - (w_1 + h_2)\left(1 - \frac{\sqrt{2+\sqrt{3}}}{2}\right) \\
&> (w_1 + h_2) + (h_1 + w_2)\sqrt{3} = |T_1|,
\end{aligned}$$

where the first inequality comes from Lemma 5 and the last inequality comes from the condition (1). ∎

4. The probability that $SMT(R) = SMT(R_1) \bigcup SMT(R_2)$

Since we have assumed before that w_1 is the largest of w_1, h_1, w_2, h_2, therefore, all h_1, w_2, h_2 will be no more than one by a further assumption $w_1 = 1$. Remember that we have assumed by symmetry that $h_1/w_1 < h_2/w_2$. It follows that $h_2 > h_1 w_2$. On these premises, the whole space of possible parameters is

$$E = \int_0^1 \int_0^1 \int_{h_1 w_2}^1 dh_2 dw_2 dh_1 = 0.75 \ .$$

To evaluate the probability that T_2 is minimal, we may assume by Theorem 2 that $h_2 > w_2$. Hence, $|T_1| = 1 + h_2 + h_1\sqrt{3} + w_2\sqrt{3}$. Let

$$\begin{aligned}
g(h_1, w_2, h_2) &= |T_2| - |T_1| \\
&= f(h_1 + h_2, 1 + w_2) + f(h_1, w_2) - 1 - h_2 - h_1\sqrt{3} - w_2\sqrt{3}. \quad (2)
\end{aligned}$$

Clearly, $g(0,0,0) = 0$, $g(0,0,h_2) < 0$ and $g(1,1,1) > 0$.

Lemma 5. $f(x + x', y + y') \leq f(x, y) + f(x', y')$.

Proof. From the triangle inequality

$$\sqrt{(x + x')^2 + (y + y')^2} \leq \sqrt{x^2 + y^2} + \sqrt{x'^2 + y'^2}$$

it follows that

$$\begin{aligned}
f(x+x',y+y') &= \sqrt{(x+x')^2+(x+x')(y+y')\sqrt{3}+(y+y')^2} \\
&= \sqrt{\Big((x+x')+\frac{\sqrt{3}}{2}(y+y')\Big)^2+\Big(\frac{1}{2}(y+y')\Big)^2} \\
&\le \sqrt{(x+\frac{\sqrt{3}}{2}y)^2+(\frac{1}{2}y)^2}+\sqrt{(x'+\frac{\sqrt{3}}{2}y')^2+(\frac{1}{2}y')^2} \\
&= \sqrt{x^2+xy\sqrt{3}+y^2}+\sqrt{x'^2+x'y'\sqrt{3}+y'^2} \\
&= f(x,y)+f(x',y').
\end{aligned}$$

■

Lemma 6. $g(h_1,w_2,h_2)$ is convex and monotonically increasing in h_1,w_2 and decreasing in h_2.

Proof. Note that

$$\frac{\sqrt{3}}{2} < \frac{\partial f}{\partial x} = \frac{2x+\sqrt{3}y}{2f(x,y)} < 1$$

and

$$\frac{\sqrt{3}}{2} < \frac{\partial f}{\partial y} = \frac{\sqrt{3}x+2y}{2f(x,y)} < 1.$$

It follows that

$$\frac{\partial g}{\partial h_1} > 0, \qquad \frac{\partial g}{\partial w_2} > 0, \qquad \frac{\partial g}{\partial h_2} < 0.$$

Moreover, by Lemma 7 we have

$$\begin{aligned}
g(\frac{h_1+h_1'}{2},\frac{w_2+w_2'}{2},\frac{h_2+h_2'}{2}) &= f(\frac{h_1+h_1'+h_2+h_2'}{2},1+\frac{w_2+w_2'}{2}) \\
&\quad + f(\frac{h_1+h_1'}{2},\frac{w_2+w_2'}{2}) \\
&\quad -1-\frac{h_2+h_2'}{2}-\frac{h_1+h_1'}{2}\sqrt{3}-\frac{w_2+w_2'}{2}\sqrt{3} \\
&\le f(\frac{h_1+h_2}{2},\frac{1+w_2}{2})+f(\frac{h_1'+h_2'}{2},\frac{1+w_2'}{2}) \\
&\quad + f(\frac{h_1}{2},\frac{w_2}{2})+f(\frac{h_1'}{2},\frac{w_2'}{2}) \\
&\quad -\frac{1}{2}(1-h_2-h_1\sqrt{3}-w_2\sqrt{3}) \\
&\quad -\frac{1}{2}(1-h_2'-h_1'\sqrt{3}-w_2'\sqrt{3}) \\
&= \frac{1}{2}\big(g(h_1,w_2,h_2)+g(h_1',w_2',h_2')\big).
\end{aligned}$$

This proves the convexity of $g(h_1, w_2, h_2)$. ∎

Now we can calculate the probability of the event that T_2 is minimal. The space of the event is $E_2 = \int\int\int_\omega dh_1 dw_2 dh_2$ where ω is bounded by $h_1 = 0, w_1 = 0, h_2 = 1$ and the surface $g(h_1, w_2, h_2) = 0$ by Lemma 8. Taking cylindrical coordinates, let $h_1 = r\cos\theta, \quad w_2 = r\sin\theta$. Hence,

$$E_2 = \int\int\int_\omega dh_2(rdr)d\theta.$$

Since we have proved that $g(h_1, w_2, h_2)$ is convex and monotonically decreasing in h_2, the interval of integration with respect to h_2 is from $h_2^*(\theta, r)$ to 1 where $h_2^*(\theta, r)$ is the root of $g(h_1, w_2, h_2) = g(\theta, r, h_2) = 0$. Put

$$p(\theta) = \cos\theta + \sin\theta, \qquad q(\theta) = \sqrt{1 + \sqrt{3}\cos\theta\sin\theta}.$$

It is easily deduced from (2) that

$$\begin{aligned} h_2^*(\theta, r) &= \frac{(r\cos\theta)^2 + (1 + r\sin\theta)^2 + \sqrt{3}r\cos\theta(1 + r\sin\theta) - (1 + \sqrt{3}rp(\theta) - rq(\theta))^2}{2(1 + \sqrt{3}rp(\theta) - rq(\theta)) - 2r\cos\theta - \sqrt{3}(1 + r\sin\theta)} \\ &= \frac{r^2(2\sqrt{3}p(\theta)q(\theta) - 3p^2(\theta)) + r((2 - 2\sqrt{3})\sin\theta - \sqrt{3}\cos\theta + 2q(\theta))}{r(\sqrt{3}\sin\theta + (2\sqrt{3} - 2)\cos\theta - 2q(\theta)) + 2 - \sqrt{3}}. \end{aligned}$$

Furthermore, the interval of integration with respect to r is from 0 to $r^*(\theta)$ where $r^*(\theta)$ is the positive root of the equation $h_2^*(\theta, r) = 1$, i.e., the quadratic equation

$$r^2(2\sqrt{3}p(\theta)q(\theta) - 3p^2(\theta)) + r\big((2 - 3\sqrt{3})p(\theta) + 4q(\theta)\big) - 2 + \sqrt{3} = 0. \tag{3}$$

Finally, the interval of integration with respect to θ is clearly from 0 to $\pi/2$. Due to the symmetry of $p(\theta), q(\theta)$ with respect to θ, equation (3) is also symmetric. Its root have extremes at $\theta = 0$ and $\theta = \pi/4$. Hence, it is easy to obtain

$$\min r^*(\theta) = r^*(\frac{\pi}{4}) = 0.241, \quad \max r^*(\theta) = r^*(0) = 0.286\ .$$

Since $g(\theta, r, h_2)$ is convex, we obtain the bounds of E_2 as

$$\begin{aligned} 0.0152 = \frac{1}{3}\cdot\frac{\pi(\min r^*)^2}{4} &= \int_0^{\pi/2}\int_0^{\min r^*(\theta)}\int_0^1 dh_2(rdr)d\theta \\ &< E_2 \\ &< \int_0^{\pi/2}\int_0^{\max r^*(\theta)}\int_0^1 dh_2(rdr)d\theta = \frac{\pi(\max r^*)^2}{4} = 0.0642\ . \end{aligned}$$

Using a mathematical software like Maple or Mathematica we get the accurate value of this integral:

$$E_2 = \int_0^{\pi/2} \int_0^{r^*(\theta)} \int_{h_2^*(\theta,r)}^1 dh_2(rdr)d\theta = 0.0241 \ .$$

Hence the probability that T_2 is minimal is $E_2/E = 0.0241/0.75 = 0.0321$.

Theorem 4. The probability of

$$SMT(R) = SMT(R_1 \cup R_2) = SMT(R_1) \bigcup SMT(R_2)$$

is $(1 - E_2/E) = 0.9679$.

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