A Neighbourhood Union Condition for Pancyclicity

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Abstract

Let G be a 2-connected graph. It is proved that if for each pair of nonadjacent vertices u and v of G, $|N(u) \cup N(v)| \geq \nu-\delta+1$, then G is vertex-pancyclic, which implies a conjecture of R. J. Faudree, R. J. Gould, M. S. Jacobson and L. Lesniak.

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All graphs considered are finite, undirected and simple.

A graph G is said to be vertex-pancyclic if for each vertex v in G, v is contained in a cycle of length m in G for each m such that $3 \le m \le \nu(G)$.

We define $NC(G) = \min \{|N(x) \cup N(y)| \mid x, y \in V(G),\ x \neq y$ and $xy \notin E(G)\}$. And we write NC for $NC(G)$ when no confusion arises. Let C be a cycle of G. Let u be a vertex in $V(C)$. We give C an orientation. Then u^+ denotes the successor of u on C in the orientation and u^- denotes the predecessor of u on C in the orientation. Let $S \subseteq V(C)$. Then $S^+ = \{ x^+ \mid x \in S \}$ and $S^- = \{ x^- \mid x \in S \}$. Let v be a vertex in $V(G) \setminus V(C)$. $N_C(v)$ denotes $N(v) \cap V(C)$. Suppose $N_C(v) \neq \emptyset$. An A-structure on $N_C(v)$ is a pair of vertices x and y such that $x, y \in N_C(v)$ and $x^+ = y$. A suc-J-structure on S is an edge x^+y^+ such that $x, y \in S$, $x^+ \neq y$ and $y^+ \neq x$. A pre-J-structure on S is an edge x^-y^- such that $x, y \in S$, $x^- \neq y$ and $y^- \neq x$. Because of the obvious similarity between suc-J-structures and pre-J-structures, for ease of notation and presentation, we frequently give proofs only using suc-J-structures (or pre-J-structures). We denote by $C^+[u,v]$ the path on C from u to v in the orientation and by $C^-[u,v]$ the path on C from u to v in the reverse orientation. The end vertices u and v are included.

Let G and H be two graphs such that $E(G) \cap E(H) = \emptyset$. We use G+H to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

For terminology and notation not defined in this paper, the reader is referred to [2].

In [3], Faudree, Gould, Jacobson and Lesniak conjectured that if G has order ν, connectivity t and satisfies $NC \ge \nu-t$ with $\delta \ge t+1$, then G is vertex-pancyclic. Song [5] reformulated their conjecture on the Chinese Symposium on Cycle Problems in Graph Theory in the form that if G is 2-connected and $NC \ge \nu-\delta+1$, then G is vertex-pancyclic. Obviously, Song's conjecture implies the conjecture by Faudree et al. In this paper, we prove Song's conjecture. The main idea of the proof is similar to that in [1].

2. The main result

Lemma 1 Let G be a 2-connected graph. If $NC \ge \nu-\delta+1$, then every vertex of G lies on a triangle. Furthermore, if $\nu \ge 4$ then every vertex of G lies on a 4-cycle.

Proof Let u be a vertex of G. If there are two distinct vertices x,y in N(u) such that $xy \in E(G)$, then uxyu is a triangle containing u. Otherwise $|N(x) \cup N(y)| \le \nu - |N(u)| \le \nu-\delta$ for any two vertices $x,y \in N(u)$, contradicting $NC \ge \nu-\delta+1$.

Now assume $\nu \geq 4$. First, we claim $\delta \geq 3$. Suppose $d(x) \leq 2$ for a vertex x in G. As $\nu(G) \geq 4$, there is a vertex y such that $xy \notin E(G)$. But $x, y \notin N(x) \cup N(y)$, which contradicts the hypothesis that $|N(x) \cup N(y)| \geq \nu-\delta+1 \geq \nu-2+1 = \nu-1$.

Assume that T = uvwu is a triangle containing u. Suppose there is no 4-cycle containing u. Since $\delta \geq 3$, there is a vertex t in $N(u)\setminus\{v,w\}$. Then $tv, tw \notin E(G)$, otherwise there is a 4-cycle containing u. Now $(N(t)\setminus\{u\}) \cap (N(w)\setminus\{u\}) = \emptyset$ and $(N(v)\setminus\{u\}) \cap (N(w)\setminus\{u\}) = \emptyset$, otherwise there is a 4-cycle containing u. So $(N(v) \cup N(t)) \cap (N(w)\setminus\{u,v\}) = \emptyset$. But $v, t \notin N(v) \cup N(t)$. Hence $|N(v) \cup N(t)| \leq \nu - |N(w)\setminus\{u,v\}| - |\{t,v\}| = \nu-d(w) < \nu-\delta+1$, contradicting the hypothesis. □

Lemma 2 Let G be a 2-connected graph with cycle C and let $u \in V(G)\setminus V(C)$. Suppose $NC \geq \nu-\delta+1$. If $|N_C(u)| \geq 2$ and there exist two distinct vertices $u_1^+, u_2^+ \in N_C^+(u)$ (or $u_1^-, u_2^- \in N_C^-(u)$) such that there is no edge from $\{u_1^+, u_2^+\}$ ($\{u_1^-, u_2^-\}$) to $N(u)\setminus V(C)$, then there is a cycle C' of length $|V(C)|+1$ such that $V(C') = V(C) \cup \{u\}$.

Proof We give C an orientation. If there is an A-structure on $N_C(u)$ such that $x, y \in N_C(u)$ and $x^+ = y$. Then $C' = C^+[y,x] + xuy$ is the desired cycle. So suppose there is no A-structure on $N_C(u)$. And then $u_1^+ \neq u_2$ and $u_2^+ \neq u_1$.

Suppose there is a suc-J-structure x^+y^+ on $N_C(u)$. Then $C' = C^+[y^+,x] + xuy + C^-[y,x^+] + x^+y^+$ is the desired cycle. Otherwise, there is no suc-J-structure on $N_C(u)$. By the hypothesis, $(N(u)\setminus V(C)) \cap (N(u_1^+) \cup N(u_2^+)) = \emptyset$. So $|N(u_1^+) \cup N(u_2^+)| \le \nu - |N_C^+(u)| - |N(u)\setminus V(C)| = \nu-d(u) < \nu-\delta+1$, contradicting $NC \ge \nu-\delta+1$. □

Theorem 3 Let G be a 2-connected graph. If $NC \ge \nu-\delta+1$, then G is vertex-pancyclic.

Proof By Lemma 1, when $\nu = 3,4$, Theorem 3 holds. Thus we may assume $\nu \ge 5$.

Suppose G is not vertex-pancyclic. Let v be a vertex of G which does not lie on any cycle of length r for some r $(3 \le r \le \nu)$. By Lemma 1, we will assume that m is the minimum number such that $3 \le m \le \nu-2$ and there is a cycle C of length m in G containing v but there is no cycle of length m+2 in G containing v. We give C an orientation.

Claim 1 For each $u \in V(G)\setminus V(C)$, there is no edge from $N_C^+(u) \cup N_C^-(u)$ to $N(u)\setminus V(C)$, for otherwise a cycle of length m+2 containing v results.

Since G is 2-connected and $m \le \nu-2$, there are two distinct vertices $x,y \in V(G)\setminus V(C)$ such that $N_C(x) \ne \emptyset$ and $N_C(y) \ne \emptyset$. Now we consider the following two cases.

Case 1 There are two distinct vertices $x,y \in V(G)\setminus V(C)$ such that $|N_C(x)| \ge 2$ and $|N_C(y)| \ge 2$.

Subcase (1.1) There are two distinct vertices x_1, $x_2 \in N_C(x)$ and two distinct vertices y_1, $y_2 \in N_C(y)$ such that $|\{x_1,x_2\} \cap \{y_1,y_2\}| \le 1$.

Subcase (1.1.1) There is an A-structure on $N_C(x)$ (or on $N_C(y)$).

If so, then there is a cycle C' of length $|V(C)|+1$ such that $V(C') = V(C) \cup \{x\}$. We give C' an orientation. By the assumption of Case (1.1), either $|N^+_{C'}(y) \cap (N^+_C(y) \cup N^-_C(y))| \geq 2$ or $|N^-_{C'}(y) \cap (N^+_C(y) \cup N^-_C(y))| \geq 2$. By Claim 1 and the argument of Lemma 2 on C', we have a cycle C'' of length $m+2$ such that $V(C'') = V(C') \cup \{y\}$, a contradiction.

Subcase (1.1.2) There is neither an A-structure on $N_C(x)$, nor an A-structure on $N_C(y)$.

Without loss of generality, assume $|N_C(x)| \leq |N_C(y)|$. By Claim 1 and Lemma 2 (and the proof of Lemma 2), there is a suc-J-structure $x_1^+x_2^+$ on $N_C(x)$ and there is a cycle $C' = C^+[x_2^+, x_1] + x_1xx_2 + C^-[x_2, x_1^+] + x_1^+x_2^+$ such that $V(C') = V(C) \cup \{x\}$. We give C' an orientation such that C' and C have the same orientation on $C^+[x_2^+, x_1]$.

If $|N_C(y) \setminus \{x_1, x_1^+, x_2, x_2^+\}| \geq 1$, then either $|N^+_{C'}(y) \cap (N^+_C(y) \cup N^-_C(y))| \geq 2$ or $|N^-_{C'}(y) \cap (N^+_C(y) \cup N^-_C(y))| \geq 2$. By Claim 1 and the argument of Lemma 2 on C', we have a cycle C'' of length $m+2$ such that $V(C'') = V(C') \cup \{y\}$, a contradiction.

So $N_C(y) \subseteq \{x_1, x_1^+, x_2, x_2^+\}$. Since there is no A-structure on $N_C(y)$, $N_C(y) = \{y_1, y_2\}$, where $y_1 \in \{x_1, x_1^+\}$ and $y_2 \in \{x_2, x_2^+\}$. But $|N_C(x)| \leq |N_C(y)|$, so $N_C(x) = \{x_1, x_2\}$. There are now two subcases.

Subcase (1.1.2.1) $x_1 = y_1$ and $x_2^+ = y_2$ (or $x_2 = y_2$ and $x_1^+ = y_1$).

By Claim 1 and Lemma 2, there is a suc-J-structure $y_1^+y_2^+$ on $N_C(y)$ and $C' = C^+[y_2^+,y_1] + y_1yy_2 + C^-[y_2,y_1^+] + y_1^+y_2^+$ is a cycle of length m+1 containing v. We give C' an orientation such that C' and C have the same orientation on $C^+[y_2^+,y_1]$. Then $|N_{C'}^-(x) \cap (N_C^+(x) \cup N_C^-(x))| \geq 2$. By Claim 1 and Lemma 2, there is a cycle C" of length m+2 containing v, a contradiction.

Subcase (1.1.2.2) $x_1^+ = y_1$ and $x_2^+ = y_2$.

Clearly, it follows that $C'' = C^+[y_2,x_1] + x_1xx_2 + C^-[x_2,y_1] + y_1yy_2$ is a cycle of length m+2 containing v, a contradiction.

Subcase (1.2) $N_C(x) = N_C(y) = \{x_1,x_2\}$.

Subcase (1.2.1) $xy \in E(G)$.

If $x_1 = x_2^+$ or $x_2 = x_1^+$, then it contradicts Claim 1. So there is no A-structure on $N_C(x)$. By Claim 1 and Lemma 2, there is a suc-J-structure $x_1^+x_2^+$ on $N_C(x)$. Then $C'' = C^+[x_2^+,x_1] + x_1xyx_2 + C^-[x_2,x_1^+] + x_1^+x_2^+$ is a cycle of length m+2 containing v, a contradiction.

Subcase (1.2.2) $xy \notin E(G)$.

Let $w \in \{x_1^+,x_2^+\}\setminus\{x_1,x_2\}$. Then by Claim 1, $(N(x) \cup N(y)) \cap (N(w)\setminus\{x_1,x_2\}) = \emptyset$. Also $x,y \notin N(x) \cup N(y)$. So $|N(x) \cup N(y)| \leq \nu - d(w) < \nu-\delta+1$, a contradiction.

Case 2 There is at most one vertex $x \in V(G)\setminus V(C)$ such that $|N_C(x)| \geq 2$.

Since $m \leq \nu-2$ and G is 2-connected, there is a vertex $y \in V(G)\setminus V(C)$ such that $|N_C(y)| = 1$. Let $N_C(y) = \{y_C\}$.

Claim 2 $y_C^-y_C^+ \in E(G)$.

Otherwise, by Claim 1, $NC \le |N(y_C^-) \cup N(y_C^+)| \le \nu - |N(y) \setminus \{y_C\}| - |\{y_C^-, y_C^+\}| \le \nu - \delta$, a contradiction.

Claim 3 $y_C^- w^+ \in E(G)$ if $y_C^- w \in E(G)$ for each $w \in V(C)$.

Suppose $y_C^- w \in E(G)$. If there is an edge from w^+ to $N(y) \setminus V(C)$, then there is a cycle C" of length m+2 containing v, a contradiction. Now suppose $y_C^- w^+ \notin E(G)$ and $N(w^+) \cap (N(y) \setminus V(C)) = \emptyset$. Then $NC \le |N(y_C^-) \cup N(w^+)| \le \nu - |N(y) \setminus \{y_C\}| - |\{y_C^-, w^+\}| \le \nu - \delta$, a contradiction, then Claim 3 follows.

Now by Claims 2 and 3, y_C^- is adjacent to all vertices on C.

Claim 4 For any two distinct vertices $u, w \in V(C) \setminus \{y_C\}$, $uw \in E(G)$.

By Claim 3, if either u or w is y_C^- then the claim follows. So $u \ne y_C^-$ and $w \ne y_C^-$. Since $y_C^- u^-$, $y_C^- w^- \in E(G)$ and there is no cycle of length m+2 containing v, it must be the case that neither u nor w is adjacent to any vertex in $N(y) \setminus \{y_C\}$. Hence $NC \le |N(u) \cup N(w)| \le \nu - |N(y) \setminus \{y_C\}| - |\{u, w\}| \le \nu - \delta$ unless $uw \in E(G)$.

Consequently the graph induced by $V(C) \setminus \{y_C\}$ is complete.

Let D be the component of G-V(C) containing y. Let P $= y_1 y_2 \ldots y_k$ be a shortest path in D with $y_1 = y$ and y_k adjacent to a vertex in $V(C) \setminus \{y_C\}$. Since $|N_C(y)| = 1$, $k \ge 2$.

If k = 2, a cycle C" of length m+2 such that $V(C") = V(C) \cup \{y_1, y_2\}$ results, a contradiction.

If $k = 3$ and if either $vy_1 \in E(G)$ or $vy_3 \in E(G)$, then it follows that v lies on a cycle of length $m+2$ which contains $\{y_1, y_2, y_3\}$ and all vertices but a vertex of C, a contradiction.

Otherwise vy_1, $vy_3 \notin E(G)$.

If $|N_C(y_1) \cup N_C(y_3)| \geq 3$, then $|N_C(y_3) \setminus \{y_C\}| \geq 2$, and v lies on a cycle of length $m+2$ which contains $\{y_1, y_2, y_3\}$ and misses a vertex in $N_C(y_3)$.

Hence we may suppose $|N_C(y_1) \cup N_C(y_3)| \leq 2$. If $|V(C)| \geq 4$, $v \in \{y_C^+, y_C^-\}$ and $N_C(y_3) \setminus \{y_C\} \subseteq \{y_C^+, y_C^-\}$, then v lies on a cycle of length $m+2$ which contains $\{y_1, y_2, y_3\}$ and misses exactly one vertex in $V(C)$, a contradiction. So suppose this is not the case. Then $N(v) \cap (N(y_1) \setminus V(C)) = \emptyset$, for otherwise the case that $k = 2$ results. Additionally, $N(v) \cap (N(y_3) \setminus V(C)) = \emptyset$, otherwise, by a similar argument as in the case that $k = 2$, v lies on a cycle of length $m+2$. Hence $NC \leq |N(y_1) \cup N(y_3)| \leq \nu - |N(v) \setminus (N_C(y_1) \cup N_C(y_3))| - |\{y_1, y_3\}| \leq \nu - d(v) \leq \nu - \delta$, a contradiction.

Suppose $k \geq 4$. Let $u \in V(D) \setminus (N(y) \cup \{y, x\})$. Since $|N_C(u)| \leq 1$, we can find a vertex $w \in V(C) \setminus \{y_C\}$ such that $uw \notin E(G)$ and $N(w) \cap (N(y) \setminus \{y_C\}) = \emptyset$. If $N(u) \cap (N(y) \setminus \{y_C\}) = \emptyset$, then $NC \leq |N(u) \cup N(w)| \leq \nu - |N(y) \setminus \{y_C\}| - |\{u, w\}| < \nu - \delta$, a contradiction. Hence $N(u) \cap (N(y) \setminus \{y_C\}) \neq \emptyset$ for all $u \in V(D) \setminus (N(y) \cup \{y, x\})$. So $k = 4$ and $x = y_4$. And x is adjacent to all vertices in $V(C) \setminus \{y_C\}$, otherwise we take a vertex w in $V(C)$ such that $xw \notin E(G)$, then

$NC \leq |N(x) \cup N(w)| \leq \nu - |N(y)\setminus\{y_C\}| - |\{x,w\}| < \nu-\delta$, a contradiction.

We now have a path $P = y_1y_2y_3y_4$ in D, where $y = y_1$ and $x = y_4$. By the assumption on P, $N_C(y_3)\setminus\{y_C\} = \emptyset$. Let $u \in V(C)\setminus\{y_C\}$.

Suppose $N(y_3) \cap (N(u)\setminus\{y_C,y_4\}) = \emptyset$. Obviously, $N(y_1) \cap (N(u)\setminus\{y_C\}) = \emptyset$. Thus $NC \leq |N(y_1) \cup N(y_3)| \leq \nu - |N(u)\setminus\{y_C,y_4\}| - |\{y_1,y_3\}| \leq \nu-d(u) < \nu-\delta+1$, a contradiction.

Finally, suppose $w \in N(y_3) \cap (N(u)\setminus\{y_C,y_4\})$. Then either $w \in N(y)\setminus\{y_C\}$ or $N(w) \cap (N(y)\setminus\{y_C\}) \neq \emptyset$. So we have a shorter path than P in D, contradicting the assumption on P. The proof of this theorem is complete. □

Remark 1 $K_{m,m}$ shows that the bound on NC in Theorem 3 is the best possible.

Remark 2 Recently the authors learned that Lin and Song [4] have obtained an analogous result for edge pancyclicity.

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