The Minimum Size of a Maximal Strong Matching in a Random Graph

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Abstract

Let $G_{n,p}$ be the random graph with fixed edge probability p, $0 < p < 1$. A strong matching S in $G_{n,p}$ is a set of vertex-disjoint edges $\{e_1, e_2, \ldots, e_m\}$ such that no other edge of the graph connects an end-vertex of e_i with an end-vertex of e_j, $e_i \neq e_j$. We show in this paper, that there exist positive constants c_1 and c_2 such that, with probability tending to 1 as $n \to \infty$, the minimum size of a maximal strong matching in $G_{n,p}$ lies between $1/2 \log_d n - c_1 \log_d \log_d n$ and $1/2 \log_d n + c_2 \log_d \log_d n$ where $d = 1/(1-p)$.

1 Introduction

Let $G_{n,p}$ denote the random graph on n vertices with edge probability p fixed, $0 < p < 1$. Throughout this paper, we set $d = 1/(1-p)$. By the expression: "*almost always*", we mean: with probability tending to 1 as $n \to \infty$.

A *strong matching* of $G_{n,p}$ is a set $\{e_1, e_2, \ldots, e_m\}$ of vertex-disjoint edges such that no other edge of the graph connects an end-vertex of e_i with an end-vertex of e_j, $i \neq j$.

In [2] we proved that, almost always, the maximum size of a strong matching in $G_{n,p}$ achieves only a finite number of values. More precisely, we established the following theorem.

Theorem 1 *There exist positive constants c_1 and c_2 depending only on p and not on n, such that:*

1) *Almost always, $G_{n,p}$ contains a strong matching of size m for each m satisfying $m \leq \log_d n - \frac{1}{2} \log_d \log_d n - c_1$.*

2) *Almost always, $G_{n,p}$ does not contain a strong matching of size m for each m satisfying $m \geq \log_d n - \frac{1}{2} \log_d \log_d n + c_2$.*

The purpose of this paper is to evaluate the minimum size of a maximal strong matching in $G_{n,p}$. We shall prove the following theorem.

Theorem 2 *There exist positive constants c_1, c_2, c_3 and c_4 depending only on p and not on n, such that :*

1) *Almost always, $G_{n,p}$ has a maximal strong matching of size m for each m satisfying $1/2\log_d n + c_3 \log_d \log_d n \leq m \leq \log_d n - \frac{1}{2}\log_d \log_d n - c_1$.*

2) *Almost always, $G_{n,p}$ has no maximal strong matching of size m for each m satisfying $m < 1/2\log_d n - c_4 \log_d \log_d n$ or $m \geq \log_d n - \frac{1}{2}\log_d \log_d n + c_2$.*

We shall make use of the following lemma concerning the tail of the binomial distribution, which can be deduced from Chernoff bounds.

Lemma 1 *Let $S_{n,p}$ denote the binomial random variable with parameters n and p. Then, for any $\epsilon > 0$ sufficiently small, we have*

$$P(|S_{n,p} - pn| \geq \epsilon pn) < 2e^{-\epsilon^2 pn/3}.$$

2 Proof of Theorem 2

Let X_m denote the number of maximal strong matchings of size m contained in $G_{n,p}$. Clearly, we have

$$E(X_m) = \binom{n}{2m}\binom{2m}{2,\ldots,2}\frac{1}{m!}\pi \sim \frac{n^{2m}}{m!2^m}\pi \tag{1}$$

where π is the probability that any fixed matching of size m is a maximal strong matching in $G_{n,p}$.

Let S be a fixed strong matching of size m. We denote by $N(S)$ the set of vertices which are not adjacent to any vertex of S. Then, one can easily verify that S is maximal if and only if $N(S)$ is either empty or an independent set.
Moreover, we observe that $|N(S)|$ is a binomial random variable with parameters $n - 2m$ and $(1-p)^{2m}$.

2.1 The case $m < \frac{1}{2}\log_d n - \alpha \log_d \log_d n$

We need to prove here that, if $m < \frac{1}{2}\log_d n - \alpha\log_d \log_d n$, where α is a positive constant which will be specified later, then $E(X_m)$ tends to 0 as $n \to \infty$.

In this case, the expectation of $|N(S)|$ satisfies

$$E(|N(S)|) = (n-2m)(1-p)^{2m} \geq (\log_d n)^{2\alpha} - o(1). \tag{2}$$

Let $\mathcal{A}$ and $\mathcal{B}$ denote respectively the events "*N(S) is stable*" and $\{(1-\epsilon)(\log_d n)^{2\alpha} \leq |N(S)| \leq n\}$. Clearly, we have

$$\pi \leq \Pr[G_{n,p} \text{ contains } S]\left(\Pr[\mathcal{A}/\mathcal{B}] + \Pr[\mathcal{B}^c]\right). \tag{3}$$

By Lemma 1 and relation (2), we have, for any $\epsilon > 0$ sufficiently small

$$\Pr[\mathcal{B}^c] < \exp\{-\frac{\epsilon^2}{3}(\log_d n)^{2\alpha}\}. \tag{4}$$

On the other hand

$$\Pr[\mathcal{A}/\mathcal{B}] = \sum \Pr[U \text{ is stable }]\Pr[N(S) = U] \tag{5}$$

where the sum is taken over all subsets U such that

$$(1-\epsilon)(\log_d n)^{2\alpha} \leq |U| \leq n.$$

If U is a fixed subset of vertices with cardinality k then

$$\Pr[U \text{ is stable }] = (1-p)^{\frac{k(k-1)}{2}}.$$

Thus, for sufficiently large n and for any $k \geq (1-\epsilon)(\log_d n)^{2\alpha}$, we have

$$\Pr[U \text{ is stable }] \leq \exp\{-\frac{\epsilon^2}{3}(\log_d n)^{2\alpha}\}.$$

Using (5) together with the last inequality, we get, for sufficiently large n

$$\Pr[\mathcal{A}/\mathcal{B}] \leq \exp\{-\frac{\epsilon^2}{3}(\log_d n)^{2\alpha}\}. \tag{6}$$

From (3), (4) and (6), we obtain

$$\pi \leq 2\left(p(1-p)^{2(m-1)}\right)^m \exp\{-\frac{\epsilon^2}{3}(\log_d n)^{2\alpha}\}, \tag{7}$$

and thus

$$E(X_m) \leq 2\frac{n^{2m}}{m!}\left(\frac{p(1-p)^{2(m-1)}}{2}\right)^m \exp\{-\frac{\epsilon^2}{3}(\log_d n)^{2\alpha}\}.$$

Finally

$$E(X_m) \leq (1-p)^{-\frac{1}{2}(\log_d n)^2(1+o(1))+O\left((\log_d n)^{2\alpha}\right)}.$$

Therefore, if $\alpha > 1$ then $E(X_m) = o(1)$, and Markov's inequality concludes the proof of this part.

2.2 The case $1/2\log_d n + \beta \log_d \log_d n \le m \le \log_d n - \frac{1}{2}\log_d \log_d n - c_1$

By using Chebyshev's inequality we shall prove here that, almost always, $G_{n,p}$ contains a maximal strong matching of size m for each m satisfying the above inequalities.
Let M_m denote the number of strong matchings of order m. Let S be any fixed strong matching of size m. Clearly,

$$E(X_m) \ \ge \ E(M_m)\Pr[|N(S)| = 0]. \tag{8}$$

As $|N(S)|$ has a bimomial distribution, we have

$$\begin{aligned}\Pr[|N(S)| = 0] &= \left(1 - (1-p)^{2m}\right)^{n-2m} \\ &\sim \exp\{-(1-p)^{2m-\log_d n}\}.\end{aligned}$$

Therefore, for any constant $\beta > 0$, we get

$$\Pr[|N(S)| = 0] = 1 - o(1).$$

On the other hand, since $E(X_m^2) \le E(M_m^2)$, we obtain

$$1 \le \frac{E(X_m^2)}{E^2(X_m)} \le \frac{E(M_m^2)}{E^2(M_m)}(1 + o(1)).$$

In [2] we have shown that $\dfrac{E(M_m^2)}{E^2(M_m)} \to 1$ as $n \to \infty$. Thus, $\dfrac{E(X_m^2)}{E^2(X_m)}$ tends also to 1 as $n \to \infty$.

Finally, the case $m > \log_d n - \dfrac{1}{2}\log_d \log_d n + c_2$ follows imediately from Theorem 1.
□

References

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