On the Weighing Matrices of Order $4n$ and Weight $4n-2$ and $2n-1$

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Abstract

We give algorithms and constructions for mathematical and computer searches which allow us to establish the existence of $W(4n, 4n-2)$ and $W(4n, 2n-1)$ for many orders $4n$ less than 4000. We compare these results with the orders for which $W(4n, 4n)$ and $W(4n, 2n)$ are known. We use new algorithms based on the theory of cyclotomy to obtain new T-matrices of order 43 and JM-matrices which yield $W(4n, 4n-2)$ for $n = 5, 7, 9, 11, 13, 17, 19, 25, 31, 37, 41, 43, 61, 71, 73, 157$.

1 Introduction

Definition 1 An *orthogonal design* A, of order n, and type $(s_1, s_2, \ldots, s_u)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$ on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$ is a square matrix of order n with entries $\pm x_k$ where each x_k occurs s_k times in each row and column such that the distinct rows are pairwise orthogonal.

In other words

$$AA^T = (s_1 x_1^2 + \ldots + s_u x_u^2) I_n$$

where I_n is the identity matrix. It is known that the maximum number of variables in an orthogonal design is $\rho(n)$, the Radon number, where for $n = 2^a b$, b odd, set $a = 4c + d, 0 \leq d < 4$, then $\rho(n) = 8c + 2^d$.

Definition 2 A *weighing matrix* $W = W(n, k)$ is a square matrix with entries 0, ± 1 having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$. The number k is called the *weight* of W. A $W(n, n)$, for $n \equiv 0 \pmod 4$, 1 or 2, whose entries are ± 1 only is called *an Hadamard matrix.* A $W(n, n-1)$ for $n \equiv 0 \pmod 4$ is equivalent to an $OD(n; 1, n-1)$ and a *skew-Hadamard matrix* or order n.

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There are a number of conjectures concerning weighing matrices:

Conjecture 1 (Weighing Matrix Conjecture) *There exists a weighing matrix* $W(4t,k)$ *for* $k \in \{1,\ldots,4t\}$.

Conjecture 2 (Skew Weighing Matrix Conjecture) *When* $n \equiv 4 \pmod 8$, *there exist a skew-weighing matrix (also written as an* $OD(n;1,k)$*) when* $k \leq n-1$, $k = a^2+b^2+c^2, a,b,c$ *integers except that* $n-2$ *must be the sum of two squares.*

Conjecture 3 *When* $n \equiv 0 \pmod 8$, *there exist a skew-weighing matrix (also written as an* $OD(n;1,k)$*) for all* $k \leq n-1$.

In Seberry and Zhang [29] weighing matrices of order $4t$ and weight $2t$ are discussed and they are found to exist for all orders less than 4000 for which Hadamard matrices are known.

Conjecture 4 (Half-Full Weighing Matrix Conjecture) *There exists a weighing matrix* $W(4t,2t)$ *for all non-negative* t.

In this paper we consider two further conjectures

Conjecture 5 (The Near Skew and Near Half-Full Conjectures) *The weighing matrices* $W(4t,4t-2)$ *and* $W(4t,2t-1)$ *exist for all non-negative* t.

Weighing matrices have long been studied because of their use in weighing experiments, see Banerjee [1] and Raghavarao [26]. Sloane and Harwit [30] survey the application of weighing matrices to improve the performance of optical instruments such as spectrometers, see also [12]. For more details and other applications of weighing matrices see [12] and [23].

Given the sequence $A = \{a_1, a_2, ..., a_n\}$ of length n the *non-periodic autocorrelation function* $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \qquad s = 0,1,...,n-1. \tag{1}$$

If $A(z) = a_1 + a_2 z + \cdots + a_n z^{n-1}$ is the associated polynomial of the sequence A, then

$$A(z)A(z^{-1}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j z^{i-j} = N_A(0) + \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}),\ z \neq 0. \tag{2}$$

Given A as above of length n the *periodic autocorrelation function* $P_A(s)$ is defined, reducing $i+s$ modulo n, as

$$P_A(s) = \sum_{i=1}^{n-1} a_i a_{i+s}, \qquad s = 0,1,...,n-1. \tag{3}$$

Base, Turyn, Golay and normal sequences are finite sequences, with zero autocorrelation function, useful in constructing orthogonal designs and Hadamard matrices [8], in communications engineering [34], in optics and signal transmission problems [10, 12], etc.
If $A = \{a_1, ..., a_n\}$, $B = \{b_1, ..., b_n\}$ are two binary $(1, -1)$ sequences of length n and

$$N_A(s) + N_B(s) = 0 \quad \text{for } s = 1, ..., n-1 \tag{4}$$

then A, B are called *Golay sequences* of length n (abbreviated GS(n)). See [7, 8, 10].

Golay sequences GS(n) exist for $n = 2^a 10^b 26^c$ (Golay numbers) where a, b, c are non-negative integers [7, 10, 11, 34].

The four sequences A, B, C, D of lengths $n+p, n+p, n, n$ with entries $+1, -1$ are called *base sequences* if:

$$\begin{aligned} N_A(s) + N_B(s) + N_C(s) + N_D(s) &= \begin{cases} 0, & s = 1, ..., n-1 \\ 4n + 2p, & s = 0 \end{cases} \\ N_A(s) + N_B(s) &= 0, \qquad s = n, ..., n+p-1. \end{aligned} \tag{5}$$

Equivalently, (5) can be replaced by

$$A(z)A(z^{-1}) + B(z)B(z^{-1}) + C(z)C(z^{-1}) + D(z)D(z^{-1}) = 4n + 2p, \qquad z \neq 0 \tag{6}$$

where $A(z)$, $B(z)$, $C(z)$, $D(z)$ are the associated polynomials. Base sequences of lengths $n+1, n+1, n, n$ are denoted by BS($2n+1$). From 6 and for $p = 1$, if we set $z = 1$ we obtain

$$a^2 + b^2 + c^2 + d^2 = 4n + 2 \tag{7}$$

where a, b, c, d are the sum of the elements of A, B, C, D respectively. BS($2n+1$) for all decompositions of $4n+2$ into four squares for $n = 1, 2, ..., 24$ are given in [4, 16, 17]. Also BS($2n+1$) for $n = 25, 26, 29$ and $n = 2^a 10^b 26^c$ (Golay numbers) are given in Yang [38].

The reader is referred to Geramita and Seberry [8] for all undefined terms.

2 Preliminary Results

We make extensive use of the book of Geramita and Seberry [8]. We quote the following theorems, giving their reference from the aforementioned book, that we use:

Lemma 1 [8, Lemma 4.11] *If there exists an orthogonal design $OD(n; s_1, s_2, \ldots, s_u)$ then there exists an orthogonal design $OD(2n; s_1, s_1, es_2, \ldots, es_u)$ where e =1 or 2.*

Lemma 2 [8, **Lemma 4.4**] *If A is an orthogonal design $OD(n; s_1, s_2, \ldots, s_u)$ on the commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_u, 0)$ then there is an orthogonal design $OD(n; s_1, s_2, \ldots, s_i + s_j, \ldots, s_u)$ and $OD(n; s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_u)$ on the $u-1$ commuting variables $(\pm x_1, \pm x_2, \ldots, \pm x_{j-1}, \pm x_{j+1}, \ldots, \pm x_u, 0)$.*

Theorem 1 [8, **Theorems 2.19 and 2.20**] *Suppose $n \equiv 0 (mod\ 4)$. Then the existence of a $W(n, n-1)$ implies the existence of a skew-symmetric $W(n, n-1)$. The existence of a skew-symmetric $W(n, k)$ is equivalent to the existence of an $OD(n; 1, k)$.*

Theorem 2 [8, **Theorems 4.49 and 2.20**] *If there exist four circulant matrices A_1, A_2, A_3, A_4 of order n satisfying*

$$\sum_{i=1}^{4} A_i A_i^T = fI$$

where f is the quadratic form $\sum_{j=1}^{u} s_j x_j^2$, then there is an orthogonal design $OD(4n; s_1, s_2, \ldots, s_u)$.

Corollary 1 *If there are four $\{0, \pm 1\}$-sequences of length n and weight w with zero periodic or non-periodic autocorrelation function then these sequences can be used as the first rows of circulant matrices which can be used in the Goethals-Seidel array to form $OD(4n; w)$ or a $W(4n, w)$. If one of the sequences is skew-type then they can be used similarly to make an $OD(4n; 1, w)$. We note that if there are sequences of length n with zero non-periodic autocorrelation function then there are sequences of length $n + m$ for all $m \geq 0$.*

3 Some Existence Results

We have four principal construction methods:

(i) constructions using conference matrices;

(ii) constructions, in even orders, using skew-Hadamard matrices;

(iii) constructions using sequences with periodic and non-periodic autocorrelation function zero, found by computer search; and

(iv) constructions using sequences found by using a combination of computer searches and cyclotomy.

Theorem 3 (Half Weight Construction) *If there exists a $W(4n, 4n-2)$ then there exists a $W(4n, 2n-1)$.*

Proof.
We arrange the rows and columns of the $W(4n, 4n-2)$ so that the $(2i, 2i)$, $(2i, 2i+1)$, $(2i+1, 2i)$ and $(2i+1, 2i+1)$ elements are all zero. This can be done as, considering any row, $2i$, we can permute the columns so the $(2i, 2i)$ and $(2i, 2i+1)$ elements are zero. We permute the rows so the $(2i+1, 2i)$ element is zero. Now, for the $2i$ and

$2i+1$ rows to be orthogonal there must be an even number of columns of these rows where both the elements in column j, $j \neq 2i$, $2i+1$, are ± 1. However there are two zeros in row $2i+1$, so the remaining zero must be in position $(2i+1, 2i+1)$.

We now rearrange the rows (columns) so row (column) $2i$ becomes row (column) i and row (column) $2i+1$ becomes row (column) $2n+i$. Now the matrix has the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where each A, B, C and D is of order $2n \times 2n$, has zero diagonal and satisfies $AB^T + CD^T = 0$. Then

$$\begin{bmatrix} \frac{1}{2}(A+C) & \frac{1}{2}(B+D) \\ \frac{1}{2}(A-C) & \frac{1}{2}(B-D) \end{bmatrix},$$

is the required $W(4n, 2n-1)$. □

Theorem 4 (Double Weight Construction) *Suppose there exists a $W(4n, 2n-i)$, with $i = 1, \ldots, n$, in the form*

$$\begin{bmatrix} X & Y \\ Z & W \end{bmatrix},$$

where X and Z ($X \wedge Z$) are disjoint and Y and W ($Y \wedge W$) are disjoint. Then, if x and y are commuting variables,

$$\begin{bmatrix} xX + yZ & xY + yW \\ yX - xZ & yY - xW \end{bmatrix}$$

is an $OD(4n; 2n-i, 2n-i)$ and hence a $W(4n, 4n-2i)$ exists.

Proof. By straightforward calculation. □

Theorem 5 (Conference Matrix Construction) *Suppose there exists a conference matrix N of order $2n$ then there exists an $OD(4n; 2, 4n-2)$, a $W(4n, 4n-2)$ and a $W(4n, 2n-1)$.*

Proof. The required $OD(4n; 2, 4n-2)$, $W(4n, 4n-2)$ and $W(4n, 2n-1)$ are

$$\begin{pmatrix} aN + bI & aN - bI \\ aN - bI & -aN - bI \end{pmatrix}, \begin{pmatrix} N & N \\ N & -N \end{pmatrix} \text{and} \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}.$$

□

Corollary 2 *Let $p \equiv 1 \pmod 4$ be a prime power, or $p = q^2$ where $q+1$ is the order of a skew-Hadamard matrix, or $p = 45$ or half any other order given by* c1 - c5 *in the list below. Then there exist $OD(2p+2; 2, 4p)$, a $W(4p+2, 2p)$ and a $W(2p+2, p)$.*

We use [32, Table 11.2] to check the existence of the required $W(4t, 4t-2)$ and $W(4t, 2t-1)$ for t, odd, $t < 1000$. The legend in the Table 5 is that a conference matrix of order $2t$ exists where indicated by c. and the construction is given by:

- c1 $2(p^r + 1)$, $p^r \equiv 1 \pmod 4$ is a prime power, [32, 9, 25, 33];
- c2 $2(h-1)^2 + 1)$, h the order of a skew-Hadamard matrix, [2];
- c3 $2(q^2(q - 2) + 1)$, $q \equiv 3 \pmod 4$, $q - 2$ a prime power; [24]
- c4 $2(5 \cdot 9^{2s+1} + 1)$, $t \geq 0$, [31];
- c5 $2(n - 1)^s + 1)$, n is the order of a conference matrix, $s \geq 2$, [33].

Theorem 6 (Skew-Hadamard Construction) *Suppose there exists an $OD(4p; 1, 4p-1)$ then there exists a $W(8p; 4p-1)$, an $OD(8p; 1, 1, 8p-2)$ and hence a $W(8p; 8p-2)$.*

Corollary 3 *If an $OD(2n; 1, 2n-1)$ exists then an $OD(2^s m; 2, 2^s m - 2)$ exists for all $s \geq 2$.*

We note $OD(4n; 1, 4n-1)$ exist for $4n = 2^t \cdot 3$, $2^t \cdot 5$, $2^t \cdot 7$, $2^t \cdot 9$, $2^t \cdot 15$, for all $t \geq 3$ ([8, 20], for $2^t \cdot 21$ when $t \geq 4$ [28]) and for $2^t \cdot 13$ when $t \geq 5$ [20].

Corollary 4 *For order $2^{s+3}n$, where $s \geq 0$ and $2^{s+3}n \leq 1000$ all $OD(2^s n; 2, 2^s n - 2)$ and hence all $W(2^s n, 2^s n - 2)$ exist.*

4 Constructions using Golay Sequences

We find some new $W(4n, 4n-2)$ and $W(4n, 2n-1)$ by using the following theorem:

Theorem 7 *Let X and Y be two Golay sequences of length g. Then*

(i) $\{X1\}, \{X-\}, \{Y0\}, \{Y0\}$ are four sequences with zero non-periodic autocorrelation function of length $g+1$ and weight $4g+2$;

(ii) $\{X11-\}, \{X--1\}, \{Y101\}, \{Y-0-\}$ are four sequences with zero non-periodic autocorrelation function of length $g+3$ and weight $4g+10$.

Hence $W(4g+4, 4g+2)$, $W(4g+4, 2g+1)$, $W(4g+12, 4g+10)$, $W(4g+12, 2g+5)$, may be constructed by using the given sequences as first rows of four circulant matrices which are using in the Goethals–Seidel array.

Corollary 5 *The near skew and near half skew conjectures are true for orders $4t$ where $t =$* 81, 83, 101, 103, 105, 107, 129, 131, 161, 163, 201, 203, 209, 211, 257, 259, 261, 263, 321, 323, 401, 403, 417, 419, 513, 515, 521, 523, 641, 643, 677, 679, 801, 803, 833 *and* 835.

Proof. Use the Golay sequences of lengths $g =$ 80, 100, 104, 128, 160, 200, 208, 256, 260, 320, 400, 416, 512, 520, 640, 676, 800, and 832. □

Example 1 $X = 1\,1\,1\,-$ and $Y = 1\,1\,-\,1$ are two Golay sequences of length 4. Now

$$\{1\ 1\ 1\ -\ 1\},\ \{1\ 1\ 1\ -\ -\},\ \{1\ 1\ -\ 1\ 0\},\ \{1\ 1\ -\ 1\ 0\}$$
$$\{1\ 1\ 1\ -\ 1\ 1\ -\},\ \{1\ 1\ 1\ -\ -\ -\ 1\},\ \{1\ 1\ -\ 1\ 1\ 0\ 1\},\ \{1\ 1\ -\ 1\ -\ 0\ -\}$$

are the required four sequences to make a $W(20,18)$ and a $W(28,26)$ respectively.

5 Notation and New Results Using Cyclotomy

Definition 3 Let x be a primitive root of $F = GF(q)$, where $q = p^\alpha = ef + 1$ is a prime power. Write $G = < x >$. The *cyclotomic classes* C_i *in* F are:

$$C_i = \{x^{es+i} : s = 0, 1, \ldots, f-1\},\ i = 0, 1, \ldots, e-1.$$

We note that the C_i are pairwise disjoint and their union is G.

For fixed i and j, the *cyclotomic number* (i, j) is defined to be the number of solutions of the equation

$$z_i + 1 = z_j\ (z_i \in C_i,\ z_j \in C_j),$$

where $1 = x^0$ is the multiplicative unit of F. That is, (i, j) is the number of ordered pairs s, t such that

$$x^{es+i} + 1 = x^{et+j}\ (0 \le s, t \le f-1).$$

Note that the number of times

$$x^{es+i} - x^{et+k} \in C_j$$

is the cyclotomic number $(k-j, i-j)$.

Notation 1 Let $A = \{a_1, a_2, \ldots, a_k\}$ be a k–set; then we will use ΔA for the collection of differences between distinct elements of A, i.e,

$$\Delta A = [a_i - a_j : i \ne j, 1 \le i, j \le k].$$

Now

$$\Delta C_i = (0,0)C_i + (1,0)C_{i+1} + (2,0)C_{i+2} + \ldots$$

and

$$\begin{aligned}\Delta(C_i - C_j) &= (0,0)C_j + (1,0)C_{j+1} + \ldots \\ &\ldots + (0,0)C_i + (1,0)C_{i+1} + \ldots \\ &\ldots + (0, i-j)C_j + (1, i-j)C_{j+1} + \ldots \\ &\ldots + (0, j-i)C_i + (1, j-i)C_{i+1} + \ldots .\end{aligned}$$

Notation 2 We use $C_a \& C_b$ to denote the adjunction of two sets with repetitions remaining. If $A = \{a, b, c, d\}$ and $B = \{b, c, e\}$, then $A \& B = [a, b, b, c, c, d, e]$. $C_a \sim C_b$ is used to denote adjunction, but with the elements of the second set becoming signed. So $A \sim B = [a, b, -b, c, -c, d, -e]$.

We define $[C_i]$ the incidence matrix of the cyclotomic class C_i by

$$c_{jk} = \begin{cases} 1, & \text{if } z_k - z_j \in C_i \\ 0, & \text{otherwise.} \end{cases}$$

As $G = C_0 \cup C_1 \cup \ldots \cup C_{e-1} = GF(p^\alpha) \setminus \{0\}$, its incidence matrix is $J_{ef+1} - I_{ef+1}$ (i.e., $\sum_{s=0}^{e-1}[C_s] = J_{ef+1} - I_{ef+1}$), and the incidence matrix of $GF(p^\alpha)$ is J_{ef+1}. Therefore, the incidence matrix of $\{0\}$ will be I_{ef+1}.

The incidence matrices of $C_a \& C_b$ and $C_a \sim C_b$ will be given by

$$[C_a \& C_b] = [C_a] + [C_b] \text{ and } [C_a \sim C_b] = [C_a] - [C_b].$$

Definition 4 A set of 4 *T-matrices* T_i, $i = 1, \ldots, 4$ of order t are four circulant or type one matrices that have entries $0, +1$ or -1 and that satisfy

(i) $T_i * T_j = 0$, $i \neq j$, ($*$ denotes the Hadamard product);

(ii) $\sum_{i=1}^{4} T_i$ is a $(1, -1)$ matrix;

(iii) $\sum_{i=1}^{4} T_i T_i^T = tI_t$; and

(iv) $t = t_1^2 + t_2^2 + t_3^2 + t_4^2$, where t_i is the row (column) sum of T_i.

Definition 5 A set of 4 *JM-matrices* A_i, $i = 1, \ldots, 4$ of order n are four circulant matrices (or matrices defined on the same abelian group) that have entries $0, +1$ or -1 and that satisfy

(i) $A_i A_j^T = A_j A_i^T$ or $A_i A_j = A_j A_i$;

(ii) $\sum_{i=1}^{4} A_i A_i^T = (4n-2)I_n$; and

(iii) $4n - 2 = a_1^2 + a_2^2 + a_3^2 + a_4^2$, where a_i is the row (column) sum of A_i.

These matrices may be used in Corollary 1 of Theorem 2 to obtain

Lemma 3 *If there exist JM-matrices of order n then there exists a $W(4n, 4n-2)$.*

Lemma 4 *There exist JM–Matrices of order $n = 3, 5, 7, \ldots, 31$.*

Proof. If A, B, C, D are base sequences of lengths n, n, $n-1$, $n-1$ then we use the sequences $A_1 = \{A\}$, $A_2 = \{B\}$, $A_3 = \{0C\}$, $A_4 = \{0D\}$ as the first rows of the corresponding circulant matrices of order n which satisfy the conditions (i), (ii) and (iii) of Definition 5, i.e. they are JM–Matrices. Since base sequences are known for $n - 1 = 1, 2, \ldots, 30$ (see [14]) we obtain the desired result. □

We now show how to use linear combinations of incidence matrices of cyclotomic classes to obtain T-matrices and JM-Matrices. We use an idea of D. Hunt and J. S. Wallis which is stated in [8] to use cyclotomy in a computer search which allowed us to extend Lemma 4.154 in [8] and to obtain new JM-matrices.

t	Squares	x	T_1, T_2, T_3, T_4
$13 = 4 \times 3 + 1$	$2^2 + 3^2 + 0^2 + 0^2$	6	$[\sim \{0\}\&C_0]$, $[C_2]$, $[C_1 \sim C_3]$, $[\emptyset]$
$19 = 6 \times 3 + 1$	$1^2 + 3^2 + 3^2 + 0^2$	3	$[\{0\}\&C_0\&C_1 \sim C_2 \sim C_3]$, $[C_4]$, $[C_5]$, $[\emptyset]$
$31 = 10 \times 3 + 1$	$2^2 + 3^2 + 3^2 + 3^2$	17	$[\sim \{0\}\&C_0\&C_1 \sim C_2]$, $[C_3]$, $[C_4\&C_6 \sim C_8]$, $[C_5\&C_7 \sim C_9]$
$37 = 12 \times 3 + 1$	$1^2 + 6^2 + 0^2 + 0^2$	35	$[\{0\}\&C_0 \sim C_1\&C_2 \sim C_3]$, $[\sim C_4\&C_7\&C_8\&C_9]$, $[C_5 \sim C_6]$, $[C_{10} \sim C_{11}]$
$41 = 8 \times 5 + 1$	$4^2 + 5^2 + 0^2 + 0^2$	7	$[\sim \{0\} \sim C_1\&C_7\&C_6]$, $[C_2 \sim C_4\&C_5]$, $[C_0 \sim C_3]$, $[\emptyset]$
$43 = 14 \times 3 + 1$	$4^2 + 3^2 + 3^2 + 3^2$	29	$[\{0\}\&C_0\&C_1 \sim C_2]$, $[C_3\&C_4 \sim C_5\&C_6 \sim C_7]$, $[C_8]$, $[\sim C_9\&C_{10}\&C_{11}\&C_{12} \sim C_{13}]$
$61 = 12 \times 5 + 1$	$6^2 + 5^2 + 0^2 + 0^2$	44	$[\{0\} \sim C_0\&C_1 \sim C_2 \sim C_3\&C_4\&C_5\&C_8]$ $[C_{10}]$, $[C_6 \sim C_7 \sim C_9\&C_{11}]$, $[\emptyset]$

Table 1: Cyclotomic results for some primes.

Lemma 5 *There exist T-matrices of order $t \in \{13, 19, 25, 31, 37, 41, 43, 61\}$.*

Table 1 gives examples indicating how the required T-matrices are constructed. This lemma was initially given for the primes $p = 13, 19, 31, 37, 41, 61$ and for the prime power $q = 25$. We extended the construction to the case $p = 43$.

We also present the first rows of the above indicated circulant T–Matrices T_1, T_2, T_3 and T_4 in Table 2.

Lemma 6 *There exist JM-matrices of order $n \in \{5, 7, 9, 11, 13, 17, 19, 25, 31, 37, 41, 43, 61, 71, 73, 157\}$.*

Use the matrices given in Table 3 as A_1, A_2, A_3, A_4 in Corollary 1 of Theorem 2. We note that the weight is $4n - 2$. All these constructions are new.

For the prime power cases ($n = 9$ and $n = 25$) we selected the fields mod $x^2 + x + 2$. Observe that for these cases the matrices are not circulant.

Note that in our cases where p is prime, the number of solutions is not influenced by the choice of the primitive root x. This is because if x and $\bar{x}$ are two primitive roots and $\bar{x} = x^t$, $gcd(t, ef) = 1$ must hold. But this means that only the cyclotomic classes C_i get "relabelled". That is, $\bar{C}_i = C_{ti \bmod e}$, because $\bar{x}^{es+i} = x^{tes+ti}$ and $gcd(t, f) = gcd(t, e) = 1$. The partition of G into classes C_i's or $\bar{C}_i$'s is the same.

Both searches, the one for T-matrices and the one for JM-matrices, went through all possible linear combinations of incidence matrices of cyclotomic classes that satisfied

t	First Rows
13	− + 0 + 00000 + 000, 0000 + 00000 + 0+, 00 + 00 + + − −00 − 0, 0000000000000
19	+ + + + −0 − + − −0 + −0 + 000−, 00000 + 0000000000 + +0, 0000000000 + 00 + 0 + 000, 0000000000000000000
31	− + −00 + 0000 − 000000 + 0 − 00 + +0 + 00000, 000 + 000000000 + 0 + 000000000000000, 0000 + 00 + + + 0000 + 0 − 0 − 0 + 0000000 − 00, 000000 + 0000 − +00000000 + 00 − 0 + −0 + +
37	+ + 0 + +00000 + 000 − 00 − 0000 − 000 + 00 − + − 000 − 0, 00 + 000 + + + −00 − 00 + −000 + 00 + 000000000 + +00, 00000 + 00000 − 0 + 00000 + 0000000 − 00000000−, 000000000000000000 − 00 + 00 − +00 + 000 − 0000
41	−0 + −00 + −00000 − +00 + 0 + +00000 + 00 − −0 + +00 + 0000, 0000 − +00 + +0 + +00000000 + 0 − 0 − 00 + 00 − 00 + 000 + +−, 0 + 00000000 + 0000 − +0 + 000 − 0 − 00 − 0000000 − 0 + 000, 00000000000000000000000000000000000000000
43	+ + +0 − 0 + 00000 + 00 − 00000000 − 0000 + 000000 + 000000, 00000 + 0 − +0 + 00000 + +00 − +000 + 0000 + 0 − 0 − 00 − 00 + 0−, 00000000000000 + 0000000000000000 + 000000000 + 0, 000 − 00000 + 0 + 0 + 0000 − −00 − +00 + + − 0000 + 0 + 00 + −000
61	+ − +00 − 0 + − − 0 − + + 0 + +0 + 0 − 0 + 00 + +0 − 0 + 0 + 0 − 00 − − − +0 + 0 + − − + − 00 + 000 + + + −00, 0000 + 000000000 + 0000 + 0000000000000000 + 000000000000 + 00000000000, 000 + 00 + 000 − 000000 − 000 + 0 − −00 + 0 − 0 − 0 − 0 − 00000 + 0 + 000000 − 0 + − + 0000 + +, 0000000000000000000000000000000000000000000000000000000000000

Table 2: The first rows of particular T–Matrices.

the condition of the four squares adding up to the right number to find solutions. In general if there was one solution there were many solutions. We shall consider inequivalence in another paper.

6 Existence of $W(4n, 4n-2)$ and $W(4n, 2n-1)$

For n odd we use Table 5 with the results noted as ci where Theorem 3, Corollary 2 has been used.

We also consider, in Table 6, existence for even numbers $2t$ where $t \leq 500$. This leaves the following values < 1000 to consider:

(i) $q = 4 \cdot t$, with $t = 1, \ldots, 125$;

(ii) $q = 8 \cdot t$, with $t = 1, \ldots, 61$;

(iii) $q = 2^s \cdot t$, with $s \geq 0$, $t = 1, \ldots, r$, where $r = \left[\frac{1000}{2^{s+2} \cdot t}\right]$ or $\left[\frac{1000}{2^{s+2} \cdot t}\right] - 1$ whichever is odd.

Thus using Theorem 6 and Corollary 4 with Table 7.1 of [32] we have

Lemma 7 *$W(4q; 4q-2)$ and $W(4q; 2q-1)$ exist for all q where $4|q$ and $4q \leq 1000$ except possibly for*

$$q = 4 \cdot 47,\ 4 \cdot 59,\ 4 \cdot 89,\ 4 \cdot 97,\ 4 \cdot 101,\ 4 \cdot 107,\ 4 \cdot 109,\ 8 \cdot 59,\ 4 \cdot 119$$

which are undecided.

n	Squares	x	A_1, A_2, A_3, A_4
$5 = 2 \times 2 + 1$	$3^2 + 3^2 + 0^2 + 0^2$	2	$[\sim \{0\} \& C_0 \& C_1], [\sim \{0\} \& C_0 \& C_1],$ $[C_0 \sim C_1], [C_0 \sim C_1]$
$7 = 2 \times 3 + 1$	$5^2 + 1^2 + 0^2 + 0^2$	3	$[\sim \{0\} \& C_0 \& C_1], [\{0\} \& C_0 \sim C_1],$ $[C_0 \sim C_1], [C_0 \sim C_1]$
$9 = 4 \times 2 + 1$	$1^2 + 1^2 + 4^2 + 4^2$	α	$[\{0\} \& C_0 \& C_1 \sim C_2 \sim C_3],$ $[\{0\} \& C_0 \& C_1 \sim C_2 \sim C_3],$ $[C_0 \& C_1 \& C_2 \sim C_3],$ $[C_0 \& C_1 \sim C_2 \& C_3]$
$11 = 5 \times 2 + 1$	$1^2 + 1^2 + 6^2 + 2^2$	2	$[\sim \{0\} \& C_0 \& C_1 \sim C_2 \& C_3 \sim C_4],$ $[\sim \{0\} \& C_0 \sim C_1 \& C_2 \& C_3 \sim C_4],$ $[\sim C_0 \& C_1 \& C_2 \& C_3 \& C_4],$ $[\sim C_0 \& C_1 \sim C_2 \& C_3 \& C_4]$
$13 = 4 \times 3 + 1$	$7^2 + 1^2 + 0^2 + 0^2$	2	$[\{0\} \& C_0 \& C_1 \& C_2 \sim C_3],$ $[\{0\} \& C_0 \& C_1 \sim C_2 \sim C_3],$ $[C_0 \sim C_1 \sim C_2 \& C_3],$ $[C_0 \sim C_1 \sim C_2 \& C_3]$
$17 = 4 \times 4 + 1$	$1^2 + 1^2 + 8^2 + 0^2$	3	$[\{0\} \& C_0 \& C_1 \sim C_2 \sim C_3],$ $[\{0\} \& C_0 \sim C_1 \sim C_2 \& C_3],$ $[C_0 \sim C_1 \& C_2 \& C_3],$ $[C_0 \sim C_1 \& C_2 \sim C_3]$
$19 = 6 \times 3 + 1$	$7^2 + 5^2 + 0^2 + 0^2$	3	$[\{0\} \& C_0 \& C_1 \& C_2 \& C_3 \sim C_4 \sim C_5],$ $[\sim \{0\} \& C_0 \& C_1 \& C_2 \& C_3 \sim C_4 \sim C_5],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \sim C_4 \& C_5],$ $[C_0 \sim C_1 \sim C_2 \sim C_3 \& C_4 \& C_5]$
$25 = 8 \times 3 + 1$	$7^2 + 7^2 + 0^2 + 0^2$	α	$[\{0\} \& C_0 \& C_1 \& C_2 \& C_3$ $\& C_4 \sim C_5 \sim C_6 \sim C_7],$ $[\{0\} \& C_0 \& C_1 \& C_2 \sim C_3$ $\sim C_4 \sim C_5 \& C_6 \& C_7],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \& C_4 \& C_5 \sim C_6 \sim C_7],$ $[C_0 \sim C_1 \& C_2 \sim C_3 \& C_4 \sim C_5 \& C_6 \sim C_7]$
$31 = 6 \times 5 + 1$	$11^2 + 1^2 + 0^2 + 0^2$	11	$[\{0\} \& C_0 \& C_1 \& C_2 \& C_3 \sim C_4 \sim C_5],$ $[\{0\} \& C_0 \& C_1 \sim C_2 \& C_3 \sim C_4 \sim C_5],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \& C_4 \sim C_5],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \& C_4 \sim C_5]$
$37 = 6 \times 6 + 1$	$1^2 + 1^2 + 12^2 + 0^2$	5	$[\{0\} \& C_0 \& C_1 \& C_2 \sim C_3 \sim C_4 \sim C_5],$ $[\{0\} \sim C_0 \& C_1 \& C_2 \sim C_3 \sim C_4 \& C_5],$ $[\sim C_0 \sim C_1 \& C_2 \& C_3 \& C_4 \& C_5],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \sim C_4 \& C_5]$
$41 = 8 \times 5 + 1$	$9^2 + 9^2 + 0^2 + 0^2$	28	$[\sim \{0\} \& C_0 \& C_1 \& C_2 \& C_3$ $\& C_4 \sim C_5 \sim C_6 \sim C_7],$ $[\sim \{0\} \& C_0 \& C_1 \& C_2 \sim C_3$ $\& C_4 \sim C_5 \sim C_6 \& C_7],$ $[C_0 \& C_1 \& C_2 \& C_3 \sim C_4 \sim C_5 \sim C_6 \sim C_7],$ $[C_0 \& C_1 \sim C_2 \& C_3 \sim C_4 \sim C_5 \& C_6 \sim C_7]$
$43 = 6 \times 7 + 1$	$13^2 + 1^2 + 0^2 + 0^2$	3	$[\sim \{0\} \& C_0 \& C_1 \& C_2 \& C_3 \sim C_4 \sim C_5],$ $[\{0\} \& C_0 \& C_1 \sim C_2 \sim C_3 \& C_4 \sim C_5],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \& C_4 \sim C_5],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \& C_4 \sim C_5]$
$61 = 12 \times 5 + 1$	$11^2 + 11^2 + 0^2 + 0^2$	6	$[\{0\} \& C_0 \& C_1 \& C_2 \& C_3 \& C_4 \& C_5$ $\& C_6 \sim C_7 \sim C_8 \sim C_9 \sim C_{10} \sim C_{11}],$ $[\{0\} \& C_0 \& C_1 \& C_2 \& C_3 \& C_4 \& C_5$ $\& C_6 \sim C_7 \sim C_8 \sim C_9 \sim C_{10} \sim C_{11}],$ $[C_0 \& C_1 \sim C_2 \& C_3 \sim C_4 \& C_5$ $\sim C_6 \sim C_7 \& C_8 \& C_9 \sim C_{10} \sim C_{11}],$ $[C_0 \sim C_1 \sim C_2 \& C_3 \sim C_4 \& C_5$ $\sim C_6 \& C_7 \& C_8 \& C_9 \sim C_{10} \sim C_{11}]$
$71 = 14 \times 5 + 1$	$9^2 + 1^2 + 10^2 + 10^2$	7	$[\sim \{0\} \& C_0 \& C_1 \& C_2 \& C_3 \& C_4 \& C_5 \& C_6$ $\& C_7 \sim C_8 \sim C_9 \sim C_{10} \sim C_{11} \sim C_{12} \sim C_{13}],$ $[\{0\} \& C_0 \& C_1 \& C_2 \& C_3 \& C_4 \sim C_5 \& C_6$ $\sim C_7 \sim C_8 \& C_9 \sim C_{10} \sim C_{11} \sim C_{12} \sim C_{13}],$ $[C_0 \& C_1 \sim C_2 \& C_3 \sim C_4 \& C_5 \& C_6$ $\& C_7 \sim C_8 \& C_9 \sim C_{10} \& C_{11} \sim C_{12} \sim C_{13}],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \& C_4 \& C_5 \& C_6$ $\& C_7 \sim C_8 \& C_9 \& C_{10} \sim C_{11} \sim C_{12} \sim C_{13}]$
$73 = 8 \times 9 + 1$	$17^2 + 1^2 + 0^2 + 0^2$	5	$[\sim \{0\} \& C_0 \& C_1 \& C_2 \& C_3$ $\& C_4 \sim C_5 \sim C_6 \sim C_7],$ $[\{0\} \& C_0 \& C_1 \& C_2 \sim C_3$ $\& C_4 \sim C_5 \sim C_6 \sim C_7],$ $[C_0 \& C_1 \sim C_2 \& C_3$ $\sim C_4 \sim C_5 \& C_6 \sim C_7],$ $[C_0 \sim C_1 \& C_2 \sim C_3$ $\sim C_4 \& C_5 \sim C_6 \& C_7]$
$157 = 12 \times 13 + 1$	$25^2 + 1^2 + 0^2 + 0^2$	53	$[\sim \{0\} \& C_0 \& C_1 \& C_2 \& C_3 \& C_4 \& C_5$ $\& C_6 \sim C_7 \sim C_8 \sim C_9 \sim C_{10} \sim C_{11}],$ $[\{0\} \& C_0 \& C_1 \& C_2 \sim C_3 \& C_4 \sim C_5$ $\sim C_6 \sim C_7 \sim C_8 \& C_9 \sim C_{10} \& C_{11}],$ $[C_0 \& C_1 \& C_2 \sim C_3 \sim C_4 \sim C_5$ $\& C_6 \sim C_7 \sim C_8 \sim C_9 \& C_{10} \& C_{11}],$ $[C_0 \& C_1 \sim C_2 \sim C_3 \sim C_4 \sim C_5$ $\& C_6 \sim C_7 \& C_8 \sim C_9 \& C_{10} \& C_{11}]$

Table 3: Cyclotomic results for some primes and prime powers.

n	First Rows
5	-++++, -++++, 0+--+, 0+--+
7	-++++++, +++-+--, 0++-+--, 0++-+--
9	++++--+--, ++++--+--, 0+++-+++-, 0++++-+-+
11	-+++----+++, -+-++--++-+, 0-++++++++-, 0-++-++-++-
13	+++++++--++-+, ++++-++--+---, 0+-+---+++-+-, 0+-+---+++-+-
17	++-+++------+++-+, ++--+-++--++-+--+, 0++-+-++++++-+-++, 0++-+---++---+-++
19	+++++-++++-++-+---+, -++++-++++-++-+---+, 0+++---+--++-+++---, 0+---+-+--++-+-+++-
25	++-++++--+---+++++++--++-, ++++-++++++-++--+--+-+-+-, 0+--+++----+-++-++-++---+, 0++++--+-+---++-++---+-+-
31	+++-+--++--+-++++--+-+++-++++++, +++-+---+--+-+-++----+++--++-++, 0++-++--++++-+--+-+-+++---+----, 0++-++--++++-+--+-+-+++---+----
37	++---+---+++++--+-++-+--+++++---+---+, +-+--+---+--++-++++++++-++--+---+--+-, 0-+++-++++--+-++++--++++-+--++++-+++-, 0++--+----++-+-+-++++-+-+-++----+--++
41	-+--+++-+++++-+-++++-+-+-++-+--+--+--++++, -+--++--+++++--++-+--+++++-++--+--++-++++, 0+---++-+++++-+-++++-+----+-+-----+--+++-, 0++---+---+++-+-+++++-----+-+---+++-+++--
43	-+++++--++-+++++++-++++--+-+----++-+++-+-++, ++-++++---+++---+--+++-++------+-+-+-++-++-, 0+-++++---+++---+--+++-++------+-+-+-++-++-, 0+-++++---+++---+--+++-++------+-+-+-++-++-
61	++-++-+--++--+++++-++++++--+-+- +-++++----+-+-----++-+++-+-+++, ++-++-+--++--+++++-++++++--+-+- +-++++----+-+-----++-+++-+-+++, 0+----+-+++++----+--++-+++--++- +-+++-++---++---+--+--++--+++-, 0++----++++++----++-+--+++--++- +-+++-++-+-+----+--+--+-+-++--
71	-++--++++-++--+--+++---++++---+-++-+ -+--+-+++-+--+++--++++++--+---+++-+, +++--++++-++------++--+++++---+-++-+ -+-++--++--+-++-++++-+++--+--------, 0++--+-+--++-++--+----++-++----+-+-+ -+-+-++++-++--++++++++-+--++-++++-+, 0++--++++++-+-+-++----++-+----+--+-+ -+-++-++++++--+++++-+++-+-----+++-+
73	-++-++-++++--++-+-+-+----+++++-++--++ +-+++-+-----++++-+++++--++-++-+-++++, +++-++-++++---+-+-+-+----+-++---+--++ +-+++-------++---++++---+--++-+-++++, 0++++++++-+-+++-+--++--++-+-++-++---- ++++--+--+-+--++--++-+---+-+--------, 0++-+---+--+---+++---++--+-+--+-++++- +----+-++-+-++--+++---+++-++-+++-+--
157	-++++-+--++---+++---+-----+++++-+----+-+ -++-+++++++-++--+-++--+++-+++--++--++++ ---+++--++-+-++--++-+++++--+-+--+--+--+ ++-++++-+---+-+++++-++++-++++--++-++--+, ++-+-+-++++---+++----++++-+---+---+--+++ --+-+-+++-+-+++------+++-+++++++++-++-- ++--+---------+---++++++---+-+---+-+-++ ---++-+++-+++-+----++++---+++----+-+-+-, 0+-+++----+--++++----++-++++---+-++-+-++ --+-+-+-++++-+--+++--++++-+++++-++-++-- -+-++--+---+--+---+-+-++--+--+-++--+--+ +-+-+----+--+------+++++-+-++-+--+-+--+, 0+--++-----+++++++-+-++-++++---+-++++-++ +-----+--+++-+--+++--++++-++-++-++-+--- -++++--+--++--+---+-+-++--+--++++-+++-- +-+------+--+------+-+-+-+---++--+-++-+

Table 4: The first rows of particular JM–Matrices.

t	How	t	How	t	How	t	How	t	How
1	c1	101	g	201	c1	301	c1	401	g
3	c1	103	g	203	g	303		403	g
5	c1	105	g	205	c1	305		405	c1
7	c1	107	g	207		307	c1	407	
9	c1	109		209	g	309	c1	409	
11	[14]	111		211	c1	311		411	c1
13	c1	113	c2	213		313	c1	413	
15	c1	115	c1	215		315		415	c1
17	[14]	117	c1	217	c1	317		417	g
19	c1	119		219		319		419	g
21	c1	121	c1	221		321	c1	421	c1
23	c3	123	c3	223		323	g	423	
25	c1	125		225	c1	325		425	
27	c1	127		227		327	c1	427	c1
29	[14]	129	c1	229	c1	329		429	c1
31	c1	131	g	231	c1	331	c1	431	
33		133		233		333		433	
35		135	c1	235		335		435	
37	c1	137		237		337	c1	437	
39		139	c1	239		339	c1	439	c1
41	c1	141	c1	241		341		441	c1
43		143		243		343		443	
45	c1	145	c1	245		345		445	
47		147	c1	247		347		447	
49	c1	149		249		349		449	
51	c1	151		251		351	c1	451	
53		153		253		353		453	
55	c1	155		255	c1	355	c1	455	
57	c1	157	c1	257	g	357		457	
59		159	c1	259	g	359		459	
61	c1	161	g	261	c1	361		461	
63	c1	163	g	263	g	363		463	
65		165		265	c1	365	c1	465	c1
67		167		267		367	c1	467	
69	c1	169	c1	269		369		469	c1
71		171		271	c1	371		471	c1
73		173		273		373		473	
75	c1	175	c1	275		375		475	
77		177	c1	277		377		477	c1
79	c1	179		279	c1	379	c1	479	
81	g	181	c1	281		381	c1	481	c1
83	g	183		283		383		483	
85	c1	185		285	c1	385	c1	485	
87	c1	187	c1	287		387	c1	487	
89		189		289	c1	389		489	c1
91	c1	191		291		391		491	
93		193		293		393		493	
95		195	c1	295		395		495	
97	c1	197		297	c1	397		497	
99	c1	199	c1	299		399	c1	499	c1

c means a conference matrix of order $2t$ exists.
g means use Golay sequences see Corollary 5.
Table 5: A $W(4t, 4t-2)$ and a $W(4t, 2t-1)$ exist.

t	How	t	How	t	How	t	How	t	How
501		601	c1	701		801	c1	901	c1
503		603		703		803	g	903	
505	c1	605		705	c1	805	c1	905	
507	c1	607	c1	707		807	c1	907	
509		609	c1	709		809		909	
511	c1	611		711		811	c1	911	
513	g	613	c2	713		813		913	
515	g	615	c1	715	c1	815		915	
517	c1	617		717	c1	817		917	
519		619	c1	719		819	c1	919	
521	g	621		721		821		921	
523	g	623		723		823		923	
525	c1	625	c1	725		825		925	c1
527		627		727	c1	827		927	
529		629		729		829	c1	929	
531	c1	631		731		831		931	c1
533		633		733		833	g	933	
535	c1	635		735		835	c1	935	
537		637		737		837		937	c1
539		639	c1	739		839		939	c1
541		641	g	741	c1	841	c1	941	
543		643	g	743		843		943	
545	c3	645	c1	745	c1	845		945	c1
547	c1	647		747	c1	847	c1	947	
549	c1	649	c1	749		849	c1	949	
551		651	c1	751		851		951	c1
553		653		753		853		953	
555	c1	655		755		855	c1	955	
557		657		757		857		957	c1
559	c1	659		759		859		959	
561		661	c1	761	c2	861	c1	961	
563		663		763		863		963	
565	c1	665		765		865		965	
567		667		767		867	c1	967	c1
569		669		769		869		969	
571		671		771		871	c1	971	
573		673		773		873		973	
575		675		775	c1	875		975	c1
577	c1	677	g	777	c1	877	c1	977	
579		679	g	779		879		979	
581		681	c1	781		881		981	
583		683		783		883		983	
585		685	c1	785		885		985	
587		687	c1	787		887		987	c1
589		689		789		889	c1	989	
591	c1	691	c1	791		891		991	
593		693		793		893		993	
595		695		795		895	c1	995	
597	c1	697		797		897		997	c1
599		699		799	c1	899		999	c1

(Table 5 continued)

q	How	q	How	q	How	q	How	q	How
2 · 1	sw	2 · 101		2 · 201		2 · 301		2 · 401	
2 · 3	sw	2 · 103		2 · 203	sy	2 · 303		2 · 403	
2 · 5	sw	2 · 105	sy	2 · 205		2 · 305		2 · 405	sy
2 · 7	sw	2 · 107		2 · 207	sy	2 · 307	sy	2 · 407	sy
2 · 9	sw	2 · 109		2 · 209		2 · 309		2 · 409	
2 · 11	sw	2 · 111	sy	2 · 211	sy	2 · 311		2 · 411	sy
2 · 13	sw	2 · 113	sy	2 · 213		2 · 313	sy	2 · 413	
2 · 15	sw	2 · 115	sy	2 · 215	sy	2 · 315	sy	2 · 415	sy
2 · 17	sw	2 · 117	sy	2 · 217	sy	2 · 317		2 · 417	sy
2 · 19	sw	2 · 119		2 · 219	sy	2 · 319		2 · 419	
2 · 21	sw	2 · 121	sy	2 · 221	sy	2 · 321	sy	2 · 421	sy
2 · 23	sw	2 · 123	sy	2 · 223		2 · 323	sy	2 · 423	
2 · 25	sw	2 · 125	sy	2 · 225		2 · 325		2 · 425	sy
2 · 27	sz	2 · 127	sy	2 · 227	sy	2 · 327	sy	2 · 427	sy
2 · 29	sz	2 · 129	sy	2 · 229		2 · 329		2 · 429	
2 · 31	sz	2 · 131	sy	2 · 231	sy	2 · 331		2 · 431	sy
2 · 33	sy	2 · 133	sy	2 · 233		2 · 333	sy	2 · 433	
2 · 35	sy	2 · 135	sy	2 · 235		2 · 335		2 · 435	
2 · 37	sy	2 · 137	sy	2 · 237	sy	2 · 337		2 · 437	sy
2 · 39	sy	2 · 139	sy	2 · 239		2 · 339	sy	2 · 439	sy
2 · 41	sy	2 · 141	sy	2 · 241	sy	2 · 341		2 · 441	
2 · 43	sy	2 · 143	sy	2 · 243	sy	2 · 343		2 · 443	
2 · 45	sy	2 · 145		2 · 245		2 · 345		2 · 445	
2 · 47		2 · 147	sy	2 · 247		2 · 347		2 · 447	sy
2 · 49	sy	2 · 149		2 · 249		2 · 349		2 · 449	
2 · 51	sy	2 · 151		2 · 251		2 · 351	sy	2 · 451	
2 · 53	sy	2 · 153		2 · 253		2 · 353		2 · 453	sy
2 · 55	sy	2 · 155	sy	2 · 255	sy	2 · 355	sy	2 · 455	
2 · 57	sy	2 · 157	sy	2 · 257		2 · 357	sy	2 · 457	
2 · 59		2 · 159	sy	2 · 259		2 · 359		2 · 459	
2 · 61	sy	2 · 161	sy	2 · 261		2 · 361		2 · 461	
2 · 63	sy	2 · 163	sy	2 · 263	sy	2 · 363	sy	2 · 463	
2 · 65	sy	2 · 165	sy	2 · 265		2 · 365	sy	2 · 465	
2 · 67	sy	2 · 167		2 · 267	sy	2 · 367	sy	2 · 467	sy
2 · 69		2 · 169		2 · 269		2 · 369		2 · 469	
2 · 71	sy	2 · 171	sy	2 · 271	sy	2 · 371	sy	2 · 471	sy
2 · 73	sy	2 · 173	sy	2 · 273	sy	2 · 373		2 · 473	
2 · 75	sy	2 · 175	sy	2 · 275		2 · 375	sy	2 · 475	
2 · 77	sy	2 · 177		2 · 277		2 · 377		2 · 477	sy
2 · 79	sy	2 · 179		2 · 279	sy	2 · 379	sy	2 · 479	
2 · 81		2 · 181	sy	2 · 281	sy	2 · 381	sy	2 · 481	
2 · 83	sy	2 · 183	sy	2 · 283		2 · 383	sy	2 · 483	sy
2 · 85	sy	2 · 185	sy	2 · 285		2 · 385		2 · 485	
2 · 87	sy	2 · 187	sy	2 · 287		2 · 387	sy	2 · 487	
2 · 89		2 · 189	sy	2 · 289		2 · 389		2 · 489	
2 · 91	sy	2 · 191		2 · 291	sy	2 · 391		2 · 491	
2 · 93	sy	2 · 193		2 · 293	sy	2 · 393	sy	2 · 493	
2 · 95	sy	2 · 195	sy	2 · 295		2 · 395	sy	2 · 495	sy
2 · 97		2 · 197	sy	2 · 297	sy	2 · 397		2 · 497	sy
2 · 99	sy	2 · 199	sy	2 · 299		2 · 399	sy	2 · 499	sy

Table 6: A skew Hadamard matrix of order $4t = 2q$ means a $W(4q, 4q-2)$ and a $W(4q, 2q-1)$ exist. Table 7.1 of [33] is used.

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