Classroom Note

Almost-Isoceles Right-Angled Triangles

C.C. Chen and T.A. Peng

Department of Mathematics
National University of Singapore
Kent Ridge, Singapore 0511

Abstract. We provide an elementary method to show that there exist infinitely many right-angled triangles with integral sides in which the lengths of the two non-hypotenuse sides differ by 1. The method also enables us to construct all such right-angled triangles recursively.

1. Introduction

There does not exist any isoceles right-angled triangle with integral sides. Does there exist a right-angled triangle with integral sides in which the lengths of the two non-hypotenuse sides differ by 1? We shall call such a triangle an *almost-isoceles right-angled* (AIRA) triangle. For an AIRA-triangle, there exist positive integers x and y such that the lengths of the sides are $x, x+1$ and y respectively with $x^2+(x+1)^2 = y^2$. We shall call the triple $(x, x+1, y)$ an *AIRA-triple.* An immediate example is the triple $(3,4,5)$ and another one is $(20,21,29)$. We would need a calculator to get the next few: $(119,120,169)$, $(696,697,985)$, $(4059,4060,5741)$, etc.. Are there infinitely many AIRA-triples? If so, is there a way to find all such triples? The answer to both questions is "yes", and one can reduce the problem to a Pell's equation (see [1], p.357) and show that there are infintely many AIRA-triples. In this note, we shall however use an elementary method to show that there are infinitely many AIRA-triples and that all such triples can be obtained recursively.

2. A Recursive Construction

Consider an AIRA-triple $(x, x+1, y)$. Thus

$$x^2 + (x+1)^2 = y^2.$$

Clearly, the problem of finding all AIRA-triangles is equivalent to finding all positive integer solutions to the following Diophantine equation:

$$2x^2 + 2x + 1 = y^2.$$

To solve this, we first write

$$4x^2 + 4x + 2 = 2y^2,$$

from which we get

$$(2x+1)^2 = 2y^2 - 1.$$

Hence $2y^2 - 1$ must be a perfect square. There exists a positive integer k with $2y^2 - 1 = (y+k)^2$. Then

$$\begin{aligned} & y^2 - 2ky - (1+k^2) = 0 \\ \Rightarrow \quad & (y-k)^2 = 2k^2 + 1. \end{aligned}$$

Again, as $2k^2+1$ must also be a perfect square, there exists another positive integer t with $2k^2 + 1 = (k+t)^2$. Then

$$\begin{aligned} & k^2 - 2tk - (t^2 - 1) = 0 \\ \Rightarrow \quad & (k-t)^2 = 2t^2 - 1. \end{aligned}$$

The above derivations suggest the following simultaneous recurrence relations:

$$(a_n - b_{n-1})^2 = 2b_{n-1}^2 + 1, \tag{I}$$

$$(b_n - a_n)^2 = 2a_n^2 - 1. \tag{II}$$

From (II), we have

$$\begin{aligned} & (b_{n-1} - a_{n-1})^2 = 2a_{n-1}^2 - 1 \\ \Rightarrow \quad & b_{n-1}^2 - 2a_{n-1}b_{n-1} = a_{n-1}^2 - 1 \\ \Rightarrow \quad & 2b_{n-1}^2 + 1 = (b_{n-1} + a_{n-1})^2, \quad \text{and so by (I)} \end{aligned}$$

$$a_n = 2b_{n-1} + a_{n-1}. \tag{III}$$

In like manner, from (I), we have

$$\begin{aligned} &(a_n - b_{n-1})^2 = 2b_{n-1}^2 + 1 \\ \Rightarrow\quad & a_n^2 - 2a_n b_{n-1} = b_{n-1}^2 + 1 \\ \Rightarrow\quad & 2a_n^2 - 1 = (a_n + b_{n-1})^2, \quad \text{and so by (II)} \\ & b_n = 2a_n + b_{n-1}. \end{aligned} \tag{IV}$$

Starting with the initial condition $a_0 = 1$ and $b_0 = 2$, the two recurrence relations (III) and (IV) will easily generate infinitely many solutions (a_n, b_n), $n = 0, 1, 2, \ldots$. We shall show via Claims 1–3 below that each a_n will be the length of the hypotenuse of an AIRA-triangle and conversely, the length of the hypotenuse of any AIRA-triangle is equal to a_n for some n. In fact, for each $n = 0, 1, 2, \ldots$, the lengths of the sides of the corresponding AIRA-triangle are $x_n, x_n + 1$ and a_{n+1}, where $(2x_n + 1)^2 = 2a_{n+1}^2 - 1$.

Claim 1. a_n, b_n and x_n are positive integers, for each $n = 0, 1, 2, \ldots$.

Proof. With the initial conditions $a_0 = 1$, $b_0 = 2$, a_n and b_n are clearly positive integers. Also, as $2a_{n+1}^2 - 1$ is an odd perfect square, x_n is also an integer. ∎

Claim 2. $(x_n, x_n + 1, a_{n+1})$ is an AIRA-triple, for each $n = 0, 1, 2, \ldots,$.

Proof. We have

$$\begin{aligned} & (2x_n + 1)^2 = 2a_{n+1}^2 - 1 \\ \Rightarrow\quad & 4x_n^2 + 4x_n + 2 = 2a_{n+1}^2 \\ \Rightarrow\quad & 2x_n^2 + 2x_n + 1 = 2a_{n+1}^2 \\ \Rightarrow\quad & x_n^2 + (x_n + 1)^2 = a_{n+1}^2. \end{aligned}$$ ∎

Claim 3. Every AIRA-triple is equal to $(x_n, x_n + 1, a_{n+1})$, for some $n = 0, 1, 2, \ldots$.

Proof. Suppose to the contrary that the claim is not valid. Let $(x, x+1, y)$ be the AIRA-triple with the smallest y which is not equal to any of the $(x_n, x_n + 1, a_{n+1})$'s. Then

$$x^2 + (x+1)^2 = y^2,$$

from which we get

$$(2x + 1)^2 = 2y^2 - 1.$$

Hence $2y^2 - 1$ is a perfect square. There exists a positive integer b with

$$(b + y)^2 = 2y^2 - 1.$$

Then

$$(y-b)^2 = 2b^2 + 1,$$

which implies that $2b^2 + 1$ is a perfect square. There exists a positive integer z with

$$(z+b)^2 = 2b^2 + 1.$$

Then $z < y$ and $(b-z)^2 = 2z^2 - 1$ and so $2z^2 - 1$ is an odd perfect square. Thus there exists a positive integer t with $(2t+1)^2 = 2z^2 - 1$, which implies that $(t, t+1, z)$ is a AIRA-triple and so by the minimality of $(x, x+1, y)$, there exists a positive integer n such that

$$(x_n, x_n + 1, a_{n+1}) = (t, t+1, z).$$

But then we have

$$\begin{aligned} & (b-z)^2 = 2z^2 - 1 \\ \Rightarrow\quad & (b - a_{n+1})^2 = 2a_{n+1}^2 - 1 \\ \Rightarrow\quad & b = b_{n+1}, \quad \text{by (II)} \\ \Rightarrow\quad & (y - b_{n+1})^2 = 2b_{n+1}^2 + 1 \\ \Rightarrow\quad & y = a_{n+2}, \quad \text{by (I)}, \end{aligned}$$

so that

$$(x_{n+1}, x_{n+1} + 1, a_{n+2}) = (x, x+1, y),$$

a contradiction. ∎

3. Numerical Computation

From the argument given in the previous section, we see that, starting from $a_0 = 1$ and $b_0 = 2$, we may apply (III), (IV) successively to obtain all the AIRA-triples. We present in the following table the first seven of these triples.

n	a_n	b_n	x_n	AIRA-triple
0	1	2	3	(3, 4, 5)
1	5	12	20	(20, 21, 29)
2	29	70	119	(119, 120, 169)
3	169	408	696	(696, 697, 985)
4	985	2378	4059	(4059, 4060, 5741)
5	5741	13860	23660	(23660, 23661, 33461)
6	33461	80782	137903	(137903, 137904, 195025)

To end the paper, we would like to point out that the two sequences a_n, and b_n actually give all the solutions to the following two Pell's equations.

$$x^2 - 2y^2 = 1, \quad \text{and}$$
$$x^2 - 2y^2 = -1,$$

with $y = b_n$ and a_n respectively.

Reference

[1] Ivan Niven, Herbert S. Zukerman and Hugh L. Montgomery, *An Introduction to the Theory of Numbers,* (Fifth Edition), Wiley, New York, 1991.

(Received 3/1/95)