Cycle Decompositions of Complete and Complete Multipartite Graphs

J.L. Ramírez-Alfonsín*
Mathematical Institute, University of Oxford
24-29 St. Giles, Oxford OX1 3LB, U.K.

Abstract

This paper is concerned with the partition of edges of the complete graph K_n and the complete multipartite graph $K_{m,\ldots,m}$ into subgraphs isomorphic to cycles. We show that K_n and $K_{m,\ldots,m}$ can be decomposed into certain families of cycles by defining a special decomposition which we call a *root path decomposition*.

1 Introduction

Let G be a graph (G may have multiple edges and loops). Let P_n and C_n be a path and a cycle with n edges respectively and let K_n be the complete graph on n vertices.

Two graphs G and H are said to be *isomorphic* (written $G \cong H$) if there are bijections $\Theta : V(G) \longrightarrow V(H)$ and $\Phi : E(G) \longrightarrow E(H)$ such that $e \in E(G)$ joins vertices $u, v \in V(G)$ if and only if edge $\Phi(e) \in E(H)$ joins vertices $\Theta(u), \Theta(v) \in V(H)$.

Let $\mathbf{H}$ be a family of graphs consisting of m_i graphs H_i for $i = 1, \ldots, l$. By an $\mathbf{H}$ *decomposition* of a graph G we mean the partition of the edges of G into $\sum_{i=1}^{l} m_i$ edge-disjoint subgraphs such that m_i of them are isomorphic to H_i for each $i = 1, \ldots, l$. We write $\langle \mathbf{H} | G \rangle$ or $\langle m_1 H_1, \ldots, m_l H_l | G \rangle$ if an $\mathbf{H}$ decomposition of G exists.

In the case when $\mathbf{H}$ consist of copies of just one graph H we write $\langle H | G \rangle$ if $\langle mH | G \rangle$ for some m, and talk of an H decomposition.

B. Alspach [1] posed the following conjecture: If n is odd and the integers $a_1, \ldots, a_n$ satisfy $a_1 + a_2 + \ldots + a_m = \frac{n(n-1)}{2}$ (if n is even and $a_1 + a_2 + \ldots + a_m = \frac{n(n-2)}{2}$), $3 \leq a_i \leq n$, does $\langle C_{a_1}, \ldots, C_{a_m} | K_n \rangle$ ($\langle C_{a_1}, \ldots, C_{a_m} | K_n - F \rangle$, where $K_n - F$ is the complete graph from which a 1-factor has been removed) where C_{a_i} is a cycle of length a_i?

*Supported by a grant from the Universidad Nacional Autónoma de México.

For an excellent and recent survey on the uniform cycle decomposition, that is, Alspach's conjecture when all the cycles have the same length (i.e. $a_i = k$ for all i), see [6] which also contains open problems.

In section 2, we prove that Alspach's conjecture is true for some families of integers a_i, in particular the following cases:

[I] $\langle C_{n-1}, C_{n-2}, C_{n-3}, \ldots, C_4, C_3, C_3 | K_n \rangle$ where n is odd,

[II] $\langle nC_n, nC_{n+1} | K_{2n+1} \rangle$ where n is odd,

[III] $\langle 2C_{2n-2}, 2C_{2n-4}, \ldots, 2C_6, 3C_4 | K_{2n} - F \rangle$ where n is even.

In order to do this we first prove that a related graph K_n' can be decomposed into certain set of paths. A similar method has been used before by R. Häggkvist [4] in the case when n is even and each of the paths is required to be a Hamilton path; also a similar method is used by B. Alspach and R. Häggkvist [2].

In section 3, we consider the decomposition of $K_{m,\ldots,m}$ into cycles. Partial results are available on this problem, for example, it is known that $\langle C_{2t} | K_{r,s} \rangle$ iff $r \equiv s \equiv 0 \pmod 2$, $r, s \geq t$ and $rs \equiv 0 \pmod{2t}$; see [8] for a proof. We shall prove that $K_{m,\ldots,m}$ can be decomposed into certain families of cycles by using the same technique as used by D.G. Hoffman, C.C. Lindner and C.A. Rodger in [5]. In particular we shall prove that:

(a) $\langle \frac{t(t-1)}{2} C_3, \frac{t(t-1)}{2} C_5, \ldots, \frac{t(t-1)}{2} C_{2m-1} | K_{\underbrace{m, \ldots, m}_{t}} \rangle$ where m and t are odd and $t, m \geq 3$.

2 Cycle decomposition of K_n

We say that K_n is *path decomposed* into $P_{i_1}, P_{i_2}, \ldots, P_{i_l}$ if $\langle P_{i_1}, P_{i_2}, \ldots, P_{i_l} | K_n \rangle$. We say that K_n is *root path decomposed* if the edges of K_n may be partitioned into paths $P_{i_1}, P_{i_2}, \ldots, P_{i_l}$ where each P_{i_j} , $1 \leq j \leq l$ starts at a different vertex; clearly a necessary condition is that $l \leq n$; we say that K_n has $n - l$ *free vertices* and we say that the root path decomposition is *complete* if $l = n$, that is K_n has no free vertices.

Lemma 2.1 *K_m has a complete root path decomposition into paths:*

(a) $P_{a_1}, P_{b_1}, P_{a_2}, P_{b_2}, \ldots, P_{a_n}, P_{b_n}$ for any non-negative integers a_i, b_i such that $a_i + b_i = m - 1$, $i = 1, 2, \ldots, n$ and $m = 2n$,

(b) $P_1, P_2, \ldots, P_{2n-2}$ with $m = 2n - 1$ and

(c) m copies of $P_{\lfloor \frac{m}{2} \rfloor}$ with m odd.

Proof: (a) Take the well known Walecki construction for Hamilton paths. Specifically, with $V(K_n) = \{0, 1, \ldots, 2n-1\}$, define $P_{2n-1}^i = (n+1, n, n+2, n-1, n+3, n-2, \ldots, 2, 0, 1) + i$ (where $(a_1, a_2, \ldots, a_m) + i = (a_1 + i, a_2 + i, \ldots, a_m + i)$, reducing sums modulo $2n$) for $i = 1, 2, \ldots, n$.

Hence, since P_{2n-1}^i begins on vertex $n+i$ and ends on vertex i, then $\langle P_{a_i}, P_{b_i} | P_{2n-1}^i \rangle$ for each $i = 1, \ldots, n$ where P_{a_i} and P_{b_i} start at the beginning and at the end of P_{2n-1}^i respectively.

(b) Remove vertex n from each path P^i_{2n-1} for $i = 1, \ldots, n$ and form paths $\hat{P}_i$ for $i = 1, \ldots, 2n-2$ as follows:

$\hat{P}_{2n-2} = P^n_{2n-1} - \{n\}$ starting at vertex 0,

$\hat{P}_{2n-3} = P^1_{2n-1} - \{n\}$ starting at vertex 1 and

$\langle \hat{P}_{2(i-1)}, \hat{P}_{2(n-i)-1} | P^i_{2n-1} - \{n\} \rangle$ for $i = 2, \ldots, n-1$ where $\hat{P}_{2(i-1)}$ and $\hat{P}_{2(n-i)-1}$ start at the beginning and at the end of P^i_{2n-1} respectively.

(c) Clearly true since all paths are graceful (see [7]). □

We are now able to prove our first theorem about the cycle decomposition of complete graphs. The following proof follows closely the ideas introduced in [4].

Theorem 2.2 *Let $P_{i_1}, \ldots, P_{i_l}$ form a root path decomposition of K_n then $\langle C_{2i_1+2}, C_{2i_1+1}, \ldots, C_{2i_l+2}, C_{2i_l+1}, \underbrace{C_3, \ldots, C_3}_{t} | K_{2n+1} \rangle$ where $t = n - l$.*

Proof: Label the vertices of K_{2n+1} with $\{X\} \cup \{y^j_i | i = 1, \ldots, n \text{ and } j = 1, 2\}$; consider a root path decomposition $P_{i_1}, \ldots, P_{i_l}$ of the complete graph K_n (with set of vertices $V(K_n) = \{Y_1, \ldots, Y_n\}$) where each path $P_{i_j} = (Y_{k_1}, \ldots, Y_{k_{i_j}+1})$ corresponds in K_{2n+1} to the graph P'_{i_j} shown in figure 1.

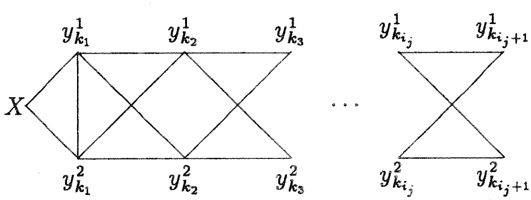

Fig.1

We say that the pair of edges $\{y^1_k, y^2_{k+1}\}$ and $\{y^2_k, y^1_{k+1}\}$ is a *cross pair* and the pair of edges $\{y^1_k, y^1_{k+1}\}$ and $\{y^2_k, y^2_{k+1}\}$ is a *straight pair.*

We decompose each P'_{i_j} into cycles C_{2i_j+2} and C_{2i_j+1} as follows: cycle C_{2i_j+2} is formed by the edges $\{X, y^1_{k_1}\}, \{X, y^2_{k_1}\}$ followed by all the crossed pair edges in P'_{i_j} up to $y^1_{k_{i_j}}$ and $y^2_{k_{i_j}}$ ending up with the edges $\{y^1_{k_{i_j}}, y^1_{k_{i_j}+1}\}$ and $\{y^2_{k_{i_j}}, y^1_{k_{i_j}+1}\}$. Similarly, cycle C_{2i_j+1} is formed by the edge $\{y^1_{k_1}, y^2_{k_1}\}$, followed by all the straight pair edges in P'_{i_j} up to $y^1_{k_{i_j}}$ and $y^2_{k_{i_j}}$ ending up with the edges $\{y^1_{k_{i_j}}, y^2_{k_{i_j}+1}\}$ and $\{y^2_{k_{i_j}}, y^2_{k_{i_j}+1}\}$.

Finally, for each of the free vertices Y_i of K_n form a triangle with the edges $\{X, y^1_i\}, \{X, y^2_i\}$ and $\{y^1_i, y^2_i\}$, $1 \leq i \leq n$. Since each P_{i_j} starts at different vertex in K_n then we have a partition of edges of K_{2n+1}. □

We can also prove a similar result for $K_{2n} - F$ using the same method but this time just asking for the existence of a path decomposition.

Let P^2_n be the graph consisting of two paths $P_x = \{x_1, \ldots, x_n\}$ and $P_y = \{y_1, \ldots, y_n\}$ where the vertices (x_i, y_{i+1}) and (y_i, x_{i+1}) for $i = 1, \ldots n-1$ are joined by an edge.

Lemma 2.3 *(R. Häggkvist [4])* P_n^2 *can be decomposed into cycles* $2C_{2q_1}, \ldots, 2C_{2q_k}$ *where* q_i *are any positive integers such that* $q_1 + q_2 + \ldots + q_k = n$ *with* $q_i \geq 2$. □

Theorem 2.4 *Let* $P_{i_1}, \ldots, P_{i_r}$ *form a path decomposition of* K_n *then* $\langle C_{P_{i_1}}, C_{P_{i_2}}, \ldots, C_{P_{i_r}} | K_{2n} - F\rangle$ *where each* $C_{P_{i_j}}$ *represent the following cycles:* $2C_{2q^1_{i_j}}$, $2C_{2q^2_{i_j}}, \ldots, 2C_{2q^m_{i_j}}$ *where* q_{i_j} *are any positive integers such that* $q^1_{i_j} + q^2_{i_j} + \ldots + q^m_{i_j} = i_j$ *with* $q^l_{i_j} \geq 2$ *for all* $1 \leq l \leq m$ *and* $1 \leq j \leq r$.

Proof: Analogous to theorem 2.2 using lemma 2.3. □

Note that the special case of theorem 2.4 in which each of the paths P_{i_j} is Hamiltonian has been proved using the same method when $n = 2p + 1$, $p \in \mathbf{N}$ in [4] and when $n = 2p$, $p \in \mathbf{N}$ in [2]. Also note that the construction used in the proof of theorem 2.2 is similar to that used in the proof of lemma 2.3.

We now give some corollaries of these two theorems.

Corollary 2.5 *(of theorem 2.2) For any integer* $n \geq 2$ *we have* $\langle C_3, C_3, C_4, C_5, \ldots, C_{2n} | K_{2n+1}\rangle$.

Proof: By theorem 2.2 it suffices to show that $P_{n-1}, P_{n-2}, \ldots, P_1$ form a root path decomposition of K_n.

[case 1] n odd. It is given in lemma 2.1 part (b).

[case 2] n even. Consider lemma 2.1 part (a) and put: $P_{a_1} = 0$, $P_{b_1} = n - 1$, $P_{a_2} = 1, P_{b_2} = n - 2, \ldots, P_{a_{\frac{n}{2}}} = \frac{n}{2} - 1, P_{b_{\frac{n}{2}}} = \frac{n}{2} + 1$. □

Corollary 2.6 *(of theorem 2.2) Let* n *be odd then* $\langle nC_n, nC_{n+1} | K_{2n+1}\rangle$.

Proof: It follows by theorem 2.2 since n is odd and by lemma 2.1 part (c) n copies of $P_{\lfloor \frac{n}{2} \rfloor}$ form a root path decomposition of K_n. □

We close this section with a corollary of theorem 2.4 close related to corollary 2 of [4].

Corollary 2.7 *(of theorem 2.4) Let* n *be any integer then* $\langle 2C_{2n-2}, 2C_{2n-4}, \ldots, 2C_6, 3C_4 | K_{2n} - F\rangle$.

Proof: It follows by theorem 2.4 and by lemma 2.1 parts (a) and (b). □

3 Cycle decomposition of $K_{m,\ldots,m}$

In this section we are interested in the cycle decomposition of the complete multipartite graph. The proof of the following theorem uses the same method as that used by Hoffman, Lindner, and Rodger in [5], (theorems 2 and 3).

Theorem 3.1 *Let* $P_{i_1}, P_{i_2}, \ldots, P_{i_r}$ *form a complete root path decomposition of* K_m *then* $\langle \frac{t(t-1)}{2} C_{2i_1+1}, \frac{t(t-1)}{2} C_{2i_2+1}, \ldots, \frac{t(t-1)}{2} C_{2i_r+1} | K_{\underbrace{m,\ldots,m}_{t}}\rangle$ *where* m *and* t *are odd and* $t \geq 3$.

Proof: Let $V(K\underbrace{m,\ldots,m}_{t}) = \{(a_i, a_j) | \ i \in \mathbf{Z}_m$ and $j \in \mathbf{Z}_t\}$. Let $\{P_{i_1}, \ldots, P_{i_r}\}$ be a root path decomposition of K_m (with set of vertices $V(K_m) = \{a_0, \ldots, a_{m-1}\}$) where each path $P_{i_j} = (a_{k_1}, \ldots, a_{k_{i_j+1}})$ corresponds in $K\underbrace{m,\ldots,m}_{t}$ to the graph $P'_{i_j}(t)$ with set of vertices $\{(a_{k_l}, a_n) | l = 1, \ldots, i_j + 1$ and $n = 0, \ldots, t-1\}$ and set of edges $\{(a_{k_l}, a_u), (a_{k_{l+1}}, a_v)\}$ for all $0 \leq u \neq v \leq t-1$ and $l = 1, \ldots, i_j$. We decompose each $P'_{i_j}(t)$ into $\frac{t(t-1)}{2}$ cycles C_{2i_j+1} as follows. For each $p = 0, \ldots, t-2$ and each $q = p+1, \ldots, t-1$, form the cycle

$(a_{k_1}, a_p), (a_{k_2}, a_q), (a_{k_3}, a_p), \ldots, (a_{k_{i_j-1}}, a_p), (a_{k_{i_j}}, a_q), (a_{k_{i_j+1}}, a_r)$
$(a_{k_{i_j}}, a_p), (a_{k_{i_j-1}}, a_q), \ldots, (a_{k_3}, a_q), (a_{k_2}, a_p), (a_{k_1}, a_q)$ if $i_j + 1$ is odd
or
$(a_{k_1}, a_p), (a_{k_2}, a_q), (a_{k_3}, a_p), \ldots, (a_{k_{i_j-1}}, a_q), (a_{k_{i_j}}, a_p), (a_{k_{i_j+1}}, a_r)$,
$(a_{k_{i_j}}, a_q), (a_{k_{i_j-1}}, a_p), \ldots, (a_{k_3}, a_q), (a_{k_2}, a_p), (a_{k_1}, a_q)$ if $i_j + 1$ is even,

where $r = r(p, q)$ corresponds to the entry a_{pq} of an idempotent symmetric latin square of order t (there always exists an idempotent symmetric latin square for odd orders, see [3]); hence, the theorem follows. □

Corollary 3.2 *For any integers m, t both odd and $m, t \geq 3$ we have* $\langle \frac{t(t-1)}{2} C_3, \frac{t(t-1)}{2} C_5, \ldots, \frac{t(t-1)}{2} C_{2m-1} | K\underbrace{m,\ldots,m}_{t} \rangle$.

Proof: It follows by theorem 3.1 and by lemma 2.1 part (b). □

The following corollary of theorem 3.1 may be found in [5] (theorem 3).

Corollary 3.3 *For any integers m, t both odd and $m, t \geq 3$ we have* $\langle C_m | K\underbrace{m,\ldots,m}_{t} \rangle$.

Proof: By lemma 2.1 part (c) K_m can be complete root path decomposed into m copies of $P_{\lfloor \frac{m}{2} \rfloor}$. Then by theorem 3.1 $\langle C_{2\lfloor \frac{m}{2} \rfloor + 1} | K_{m,\ldots,m} \rangle$ or equivalently $\langle C_m | K_{m,\ldots,m} \rangle$. □

Acknowledgment
I am grateful to Dr. Colin McDiarmid for many helpful and stimulating conversations. I also thank the referee for suggesting improvements in the presentation.

References

[1] B. Alspach, Research problems, Problem 3, *Discrete Mathematics* 36 (1981), 333.

[2] B. Alspach and R. Häggkvist, Some observations on the Oberwolfach problem, *Journal of Graph Theory* 9, (1985), 177-187.

[3] J. Dénes and A.D. Keedwell, Latin Squares and their Applications, Academic Press, New York and London, (1974).

[4] R. Häggkvist, A lemma on cycle decompositions, *Ann. of Discrete Mathematics* 27 (1985), 227-232.

[5] D.G. Hoffman, C.C. Lindner and C.A. Rodger, On the construction of odd cycles systems, *Journal of Graph Theory* 13 No.4 (1989) 417-426.

[6] C.C. Lindner and C.A. Rodger, Decomposition into Cycles II: Cycle systems, contemporary design theory, A collection of Surveys, Edited by H. Dinitz, D.R. Stinson, John Wiley and Sons, Inc. (1992), 335-369.

[7] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs*, (International Symposium, Rome July 1966), Gordon and Breach, New York, N.Y. and Dunod Paris (1967), 349-355.

[8] D. Sotteau, Decomposition of $K_{m,n}$ into circuits of length $2k$, *Journal of Combinatorial Theory (B)*, 30 (1981), 185-191.

(Received 1/5/94; revised 1/11/94)