Lambda-fold cube decompositions

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ABSTRACT: Necessary and sufficient conditions on n and λ are given for the existence an edge-disjoint decomposition of λK_n into copies of the graph of a 3-dimensional cube. Also, necessary and sufficient conditions on m, n and λ are given for similar decompositions of $\lambda K_{m,n}$.

1 Introduction

Let G and H be graphs. A *G-decomposition* of H is a set $\{G_1, G_2, \ldots, G_t\}$ of edge-disjoint subgraphs of H, each of which is isomorphic to G, such that the edge sets of the G_i's partition the edge set of H.

Necessary and sufficient conditions for a G-decomposition of H have been established for various G and H. The most common problem considered is: given a graph G, for which n does there exist a G-decomposition of K_n, the complete graph of order n. Other common choices for H include the lambda-fold complete graph λK_n, and (when G is bipartite) the λ-fold complete bipartite graph $\lambda K_{m,n}$. G-decompositions of the above graphs have been considered for many different graphs G. In this paper, we consider G-decompositions when G is the graph of the 3-dimensional cube. Throughout this paper we shall we shall use C to denote this graph (see Figure 1).

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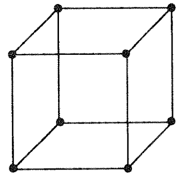

Figure 1: The graph C of the 3-dimensional cube

In 1979, Kotzig [5] found a C-decomposition of K_{16}, and posed the problem of finding necessary and sufficient conditions on n for the existence of a C-decomposition of K_n. In 1981 [6], he proved that if there exists C-decomposition of K_n then n is necessarily equivalent to 1 or 16 modulo 24 and he also proved sufficiency for the case $n \equiv 1 \pmod{24}$. The problem of showing that there exists a C-decomposition of K_n when $n \equiv 16 \pmod{24}$ was again mentioned in 1985 by Harary and Robinson [4] and was recently solved (in 1994) [2]. Necessary and sufficient conditions on n and m for the existence of a C-decomposition of $K_{m,n}$ were also given in [2].

In this paper, we give necessary and sufficient conditions (on λ and n) for the existence of a C-decomposition of λK_n and (on λ, n and m) for the existence of a C-decomposition of $\lambda K_{n,m}$. Necessary and sufficient conditions for a G-decomposition of λK_n have already been given for the graphs of two other regular solids. A G-decomposition of λK_n where G is the graph of the tetrahedron (that is, K_4) is a $(v, 4, \lambda)$ BIBD with $v = n$. Necessary and sufficient conditions for the existence of $(v, 4, \lambda)$ BIBD's are well known; see [3]. The problem of finding G-decompositions of λK_n where G is the graph of the octahedron (equivalent to a Pasch configuration) was recently solved by Adams *et al* [1].

2 3-Cube Decompositions of $\lambda K_{m,n}$

In this section, we consider C-decompositions of the λ-fold complete bipartite graph $\lambda K_{m,n}$ (where we assume $m \leq n$). We note that the case $\lambda = 1$ was done in [2]. Since C is a 3-regular bipartite graph with 12 edges and 4 vertices in each subset of the bipartition, the necessary conditions for the existence of such decompositions include:

(2.1) $3 \mid \lambda m$ and $3 \mid \lambda n$;
(2.2) $12 \mid \lambda mn$;
(2.3) $4 \leq m \leq n$.

We shall show that the above necessary conditions (2.1)-(2.3) are sufficient. We shall make frequent use of the following two simple lemmas.

Lemma 2.1 *If there are G-decompositions of K_{m,n_1} and K_{m,n_2} then there is a G-decomposition of $K_{am,b_1n_1+b_2n_2}$ for any non-negative integers a, b_1, and b_2.*

Proof. First decompose $K_{am,b_1n_1+b_2n_2}$ into ab_1 copies of K_{m,n_1} and ab_2 copies of K_{m,n_2}. Then decompose each of these into copies of G. □

Lemma 2.2 *Let H be a graph and suppose there is a G-decomposition of $\lambda_1 H$ and $\lambda_2 H$ ($\lambda_1, \lambda_2 \geq 1$). Then there is a G-decomposition of $(a\lambda_1 + b\lambda_2)H$ for any non-negative integers a and b.*

Proof. First decompose $(a\lambda_1 + b\lambda_2)H$ into a copies of $\lambda_1 H$ and b copies of $\lambda_2 H$. Then decompose each of these into copies of G. □

To establish the sufficiency of conditions (2.1)-(2.3), we shall use lemmas 2.1 and 2.2, and present all of the necessary decompositions of $\lambda K_{m,n}$.

Theorem 2.3 *For $m \leq n$, a C-decomposition of $\lambda K_{m,n}$ exists if and only if $\lambda m \equiv \lambda n \equiv 0$ (mod 3), $\lambda mn \equiv 0$ (mod 4) and $m \geq 4$.*

Proof. The necessary conditions are obtained from (2.1)-(2.3). The proof of sufficiency consists of six cases that depend on the value of the greatest common divisor of 12 and λ.

Case 1: $\gcd\{12, \lambda\} = 1$.
In this case, the necessary conditions reduce to:

$3 \mid m$ and $3 \mid n$;
$4 \mid mn$;
$4 \leq m \leq n$.

Under these conditions, either

$m \equiv n \equiv 0$ (mod 6) or
$m \equiv 0$ (mod 12) and $n \equiv 3$ (mod 6) (or vice versa).

In either case, $\lambda K_{m,n}$ can be decomposed into into a collection of graphs each of which is isomorphic to either $K_{6,6}$ or $K_{9,12}$. Thus it suffices to find C-decompositions of $K_{6,6}$ and of $K_{9,12}$. These decompositions exist; see [2].

Case 2: $\gcd\{12, \lambda\} = 2$.
In this case, the necessary conditions (2.1)-(2.3) reduce to:

$3 \mid m$ and $3 \mid n$;
$2 \mid mn$;
$4 \leq m \leq n$.

Under these conditions, either

$m \equiv n \equiv 0$ (mod 6) or
$m \equiv 0$ (mod 6) and $n \equiv 3$ (mod 6) (or vice versa).

In either case, $\lambda K_{m,n}$ can be decomposed into a collection of graphs each of which is isomorphic to either $2K_{6,6}$ or $2K_{6,9}$. A C-decomposition of $2K_{6,9}$ is given in the appendix.

Case 3: $\gcd\{12, \lambda\} = 3$.
In this case, conditions (2.1)-(2.3) reduce to:

$4 \mid mn$;
$4 \leq m \leq n$.

The needed new decompositions needed in this case are C-decompositions of $3K_{4,4}$, $3K_{4,5}$,$3K_{4,6}$, and $3K_{4,7}$. These are given in the appendix.

Case 4: $\gcd\{12, \lambda\} = 4$.
In this case, conditions (2.1)-(2.3) reduce to:

$3 \mid m$ and $3 \mid n$;

$4 \leq m \leq n$.

Here the only new case is a C-decomposition of $4K_{9,9}$, which is given in the appendix.

Case 5: $\gcd\{12, \lambda\} = 6$.
In this case, conditions (2.1)-(2.3) reduce to:

$2 \mid mn$;

$4 \leq m \leq n$.

Here the only previously uncovered small cases are C-decompositions of $6K_{5,6}$ and $6K_{6,7}$. These are given in the appendix.

Case 6: $\gcd\{12, \lambda\} = 12$.
In this case, conditions (2.1)-(2.3) reduce to:

$4 \leq m \leq n$.

The uncovered small cases are 3−cube decompositions of $12K_{5,5}$, $12K_{5,7}$ and $12K_{7,7}$. These too are given in the appendix. □

3 3-Cube Decompositions of λK_n

Lemma 3.1 *If there is a C-decomposition of λK_{x+1} then there is a C-decomposition of $\lambda K_{x,24}$.*

Proof. By Theorem 2.3, we need only show that the necessary conditions (2.1)-(2.3) are satisfied. Since there is a C-decomposition of λK_{x+1}, $x \equiv 0 \pmod 3$ and hence (noting that $x > 4$) the necessary conditions are satisfied. □

Lemma 3.2 *Let $k \geq 0$ and $n = 24k + x + 1$, where $8 \leq x + 1 < 32$. Then if there is a C-decomposition of λK_{x+1} there is a C-decomposition of λK_n.*

Proof. First we note that for all λ, there are C-decompositions of $\lambda K_{24,24}$ and λK_{25} (using Lemma 3.1 and the C-decompositions of $K_{24,24}$ and K_{25} given in [2]). Also, by Lemma 3.1, there is a C-decomposition of $\lambda K_{x,24}$.

Now, let $V(\lambda K_n) = V_0 \cup V_1 \cup \ldots \cup V_k \cup \{\infty\}$ where $V_0 = \{0_1, 0_2, \ldots, 0_x\}$ and for $i = 1, 2, \ldots, k$, $V_i = \{i_1, i_2, \ldots, i_{24}\}$. For $i = 0, 1, \ldots, k$, let G_i be the λ-fold complete graph with vertex set $V_i \cup \{\infty\}$ and for each i, j with $0 \leq i < j \leq k$, let $G_{i,j}$ be the λ-fold complete bipartite graph with vertex set $V_i \cup V_j$ (and the obvious bipartition). Then, λK_n is the edge disjoint union

$$\lambda K_n = (\cup_{0 \leq i \leq k} G_i) \bigcup (\cup_{0 \leq i < j \leq k} G_{i,j}).$$

Clearly:

(1) $G_0 \cong \lambda K_{x+1}$;

(2) for $i = 1, 2, \ldots, k$, $G_i \cong \lambda K_{25}$;

(3) for $j = 1, 2, \ldots, k$, $G_{0,j} \cong \lambda K_{x,24}$;

(4) for all i, j satisfying $1 \leq i < j \leq k$, $G_{i,j} \cong \lambda K_{24,24}$.

Hence the union of the C-decompositions of each of these subgraphs (the G_i's and the $G_{i,j}$'s) is a C-decomposition of λK_n. □

Theorem 3.3 *There is a C-decomposition of λK_n if and only if $n \geq 8$, $3 \mid \lambda(n-1)$ and $24 \mid \lambda n(n-1)$.*

Proof. The necessary conditions are established by noting that since C has degree 3, 3 must divide the degree of λK_n (that is, $3 \mid \lambda(n-1)$) and that since C has 12 edges, 12 must divide the number of edges in λK_n (that is, $12 \mid \lambda\frac{n(n-1)}{2}$).

For sufficiency, as before, the proof consists of 6 cases that depend on the value of gcd(12,λ).

Case 1: $\gcd\{12, \lambda\} = 1$.

In this case, the necessary conditions are $n \equiv 1, 16 \pmod{24}$. Hence by Lemma 3.2 we only need a C-decomposition of K_{16}, which is given in [2].

Case 2: $\gcd\{12, \lambda\} = 2$.

In this case, the necessary conditions are $n \equiv 1, 4, 13, 16 \pmod{24}$, $n \neq 4$. Hence by Lemma 3.1 and Lemma 3.2, the only new C-decompositions needed are of $2K_{28}$ and $2K_{13}$, both of which are given in the appendix.

Case 3: $\gcd\{12, \lambda\} = 3$.

In this case, the necessary conditions are $n \equiv 0, 1, 8, 9, 16, 17 \pmod{24}$. Hence by Lemma 3.1 and Lemma 3.2, the only new C-decompositions needed are of $3K_{24}, 3K_8$, $3K_9$ and $3K_{17}$, all of which are given in the appendix.

Case 4: $\gcd\{12, \lambda\} = 4$.

In this case, the necessary conditions are $n \equiv 1, 4, 7, 10, 13, 16, 19, 22 \pmod{24}$, $n \neq 4, 7$. Hence by Lemma 3.1 and Lemma 3.2, the only new C-decompositions needed are of $4K_{31}, 4K_{10}, 4K_{19}$ and $4K_{22}$, all of which are given in the appendix.

Case 5: $\gcd\{12, \lambda\} = 6$.

In this case, the necessary conditions are $n \equiv 0, 1, 4, 5, 8, 9, 12, 13, 16, 17,$ $20, 21 \pmod{24}$, $n \neq 4, 5$. Hence by Lemma 3.1 and Lemma 3.2, the only new C-decompositions needed are of $6K_{29}, 6K_{12}, 6K_{20}$ and $6K_{21}$, all of which are given in the appendix.

Case 6: $\gcd\{12, \lambda\} = 12$.

In this case, the only necessary condition is that $n \neq 2, 3, 4, 5, 6, 7$. Hence by Lemma 3.1 and Lemma 3.2, the only new C-decompositions needed are of $12K_{26}, 12K_{27}, 12K_{30}$, $12K_{11}, 12K_{14}, 12K_{15}, 12K_{18}$ and $12K_{23}$, all of which are given in the appendix. □

4 Conclusions

In this section we summarise (in tabular form) the results of this paper, see Theorems 2.3 and 3.3.

λ (mod 12)	admissible m, n $(4 \leq m \leq n)$
1, 5, 7, 11	$m, n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 4$
2, 10	$m, n \equiv 0 \pmod 3$, $mn \equiv 0 \pmod 2$
3, 9	$mn \equiv 0 \pmod 4$
4, 8	$m, n \equiv 0 \pmod 3$
6	$mn \equiv 0 \pmod 2$
12	any m, n

Table 1: Necessary and sufficient conditions for the existence of a C-decomposition of $\lambda K_{m,n}$

λ (mod 12)	admissible $n \geq 8$
1, 5, 7, 11	$n \equiv 1, 16 \pmod{24}$
2, 10	$n \equiv 1, 4 \pmod{12}$
3, 9	$n \equiv 0, 1 \pmod 8$
4, 8	$n \equiv 1 \pmod 3$
6	$n \equiv 0, 1 \pmod 4$
12	all n

Table 2: Necessary and sufficient conditions for the existence of a C-decomposition of λK_n

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5 Appendix

Within this appendix, each cube decomposition of a graph G is given as (V, C), where V is the vertex set of G, and C is the collection of cubes. The graph of the cube with vertex set $\{a, b, c, d, e, f, g, h\}$ and edge set $\{ab, bc, cd, da, ef, fg, gh, he, ae, bf, cg, dh\}$ is denoted by the 8-tuple (a, b, c, d, e, f, g, h).
The vertex set of $\lambda K_{m,n}$ is $(\mathbf{Z}_m \times \{0\}) \bigcup (\mathbf{Z}_n \times \{1\})$ (with the obvious bipartition) and the ordered pair (x, y) of this vertex set is represented by x_y.

$\boxed{\lambda = 2}$

$\boxed{2K_{6,9}}$ $V = \{i_1 \mid 0 \le i \le 5\} \cup \{i_2 \mid 0 \le i \le 8\}$ C as follows, uncycled:

$$(0_1, 0_2, 1_1, 1_2, 2_2, 2_1, 3_2, 3_1),\ (0_1, 0_2, 1_1, 1_2, 2_2, 2_1, 3_2, 3_1),$$
$$(0_1, 3_2, 4_1, 4_2, 5_2, 5_1, 0_2, 3_1),\ (0_1, 3_2, 4_1, 4_2, 5_2, 5_1, 0_2, 3_1),$$
$$(0_1, 6_2, 1_1, 7_2, 8_2, 2_1, 4_2, 5_1),\ (0_1, 6_2, 1_1, 8_2, 7_2, 2_1, 5_2, 4_1),$$
$$(1_1, 2_2, 4_1, 5_2, 4_2, 5_1, 1_2, 2_1),\ (1_1, 2_2, 4_1, 7_2, 8_2, 5_1, 6_2, 3_1),$$
$$(2_1, 1_2, 4_1, 8_2, 7_2, 5_1, 6_2, 3_1).$$

$\boxed{2K_{13}}$ $V = Z_{13}$. C as follows, cycled modulo 13:

$$(0, 1, 2, 4, 3, 8, 6, 9).$$

$\boxed{2K_{28}}$ $V = \{i_j \mid 0 \le i \le 6;\ j = 1, 2, 3, 4\}$. C as follows, cycled modulo $(7, -)$:

$$(0_1, 1_1, 2_1, 4_1, 3_1, 0_2, 1_2, 4_2),\ (0_1, 2_1, 0_2, 1_2, 2_2, 4_2, 4_1, 0_3),$$
$$(0_1, 3_2, 5_1, 0_3, 4_2, 1_2, 1_3, 2_3),\ (0_1, 0_3, 1_1, 2_3, 1_3, 2_1, 6_3, 0_2),$$
$$(0_1, 4_3, 0_2, 0_4, 1_4, 1_2, 1_3, 4_2),\ (0_1, 0_4, 1_1, 2_4, 3_4, 1_2, 4_4, 5_1),$$
$$(0_1, 2_4, 0_2, 4_4, 5_4, 0_3, 2_3, 1_3),\ (0_2, 3_3, 0_3, 1_4, 5_4, 6_3, 0_4, 2_3),$$
$$(0_2, 1_4, 1_3, 5_4, 6_4, 4_4, 3_4, 0_4).$$

$\boxed{2K_{37}}$ $V = Z_{37}$. C as follows, cycled modulo 37:

$$(0, 1, 2, 4, 3, 5, 8, 13),\ (0, 5, 11, 18, 7, 15, 2, 26),\ (0, 11, 23, 12, 14, 28, 7, 29).$$

$\lambda = 3$

$3K_{4,4}$ $V = \{i_1 \mid 0 \le i \le 3\} \cup \{i_2 \mid 0 \le i \le 3\}$ C as follows, uncycled:

$(0_1, 0_2, 1_1, 1_2, 2_2, 2_1, 3_2, 3_1)$, $(0_1, 0_2, 1_1, 1_2, 3_2, 3_1, 2_2, 2_1)$,
$(0_1, 0_2, 2_1, 2_2, 3_2, 3_1, 1_2, 1_1)$, $(0_1, 1_2, 2_1, 3_2, 2_2, 3_1, 0_2, 1_1)$.

$3K_{4,5}$ $V = \{i_1 \mid 0 \le i \le 3\} \cup \{i_2 \mid 0 \le i \le 4\}$ C as follows, uncycled:

$(0_1, 0_2, 1_1, 1_2, 2_2, 2_1, 3_2, 3_1)$, $(0_1, 0_2, 1_1, 1_2, 3_2, 3_1, 4_2, 2_1)$,
$(0_1, 0_2, 2_1, 2_2, 4_2, 3_1, 1_2, 1_1)$, $(0_1, 1_2, 2_1, 4_2, 3_2, 3_1, 2_2, 1_1)$,
$(0_1, 2_2, 1_1, 3_2, 4_2, 3_1, 0_2, 2_1)$.

$3K_{4,6}$ $V = \{i_1 \mid 0 \le i \le 3\} \cup \{i_2 \mid 0 \le i \le 5\}$ C as follows, uncycled:

$(0_1, 0_2, 1_1, 1_2, 2_2, 2_1, 3_2, 3_1)$, $(0_1, 0_2, 1_1, 1_2, 3_2, 3_1, 2_2, 2_1)$,
$(0_1, 0_2, 2_1, 2_2, 4_2, 3_1, 5_2, 1_1)$, $(0_1, 1_2, 2_1, 3_2, 5_2, 3_1, 4_2, 1_1)$,
$(0_1, 2_2, 1_1, 4_2, 5_2, 3_1, 0_2, 2_1)$, $(0_1, 3_2, 1_1, 5_2, 4_2, 3_1, 1_2, 2_1)$.

$3K_{4,7}$ $V = \{i_1 \mid 0 \le i \le 3\} \cup \{i_2 \mid 0 \le i \le 6\}$ C as follows, uncycled:

$(0_1, 0_2, 3_1, 1_2, 6_2, 1_1, 2_2, 2_1)$, $(0_1, 0_2, 3_1, 1_2, 5_2, 2_1, 3_2, 1_1)$,
$(0_1, 0_2, 2_1, 2_2, 6_2, 1_1, 1_2, 3_1)$, $(0_1, 1_2, 2_1, 3_2, 5_2, 1_1, 4_2, 3_1)$,
$(0_1, 2_2, 3_1, 4_2, 3_2, 1_1, 0_2, 2_1)$, $(0_1, 2_2, 2_1, 5_2, 4_2, 1_1, 6_2, 3_1)$,
$(0_1, 3_2, 3_1, 6_2, 4_2, 1_1, 5_2, 2_1)$.

$3K_8$ $V = Z_8$. C as follows, uncycled:

$(0, 1, 2, 3, 4, 5, 6, 7)$, $(0, 1, 2, 3, 4, 5, 6, 7)$, $(0, 1, 2, 3, 5, 4, 7, 6)$,
$(0, 2, 4, 6, 5, 7, 1, 3)$, $(0, 2, 4, 6, 7, 5, 3, 1)$, $(0, 2, 4, 6, 7, 5, 3, 1)$,
$(0, 4, 1, 5, 7, 3, 6, 2)$.

$3K_9$ $V = Z_9$. C as follows, cycled modulo 9:

$$(0, 1, 2, 3, 5, 8, 4, 7).$$

$3K_{17}$ $V = Z_{17}$. C as follows, cycled modulo 17:

$$(0,1,2,3,4,6,8,10), \quad (0,3,6,10,8,15,11,2).$$

$3K_{24}$ $V = Z_{23} \cup \{\infty\}$. C as follows, cycled modulo 23:

$$(0,1,2,3,4,6,8,10), \quad (0,3,6,10,4,9,1,17), \quad (0,8,18,9,11,22,7,\infty).$$

$3K_{32}$ $V = Z_{31} \cup \{\infty\}$. C as follows, cycled modulo 31:

$(0,1,2,3,4,6,8,10)$, $(0,3,6,10,4,9,1,17)$, $(0,7,16,8,9,18,1,19)$, $(0,10,23,11,12,24,7,\infty)$.

$3K_{33}$ $V = Z_{33}$. C as follows, cycled modulo 33:

$(0,1,2,3,4,6,8,10)$, $(0,3,6,10,4,9,1,17)$, $(0,7,16,8,9,18,1,19)$, $(0,10,23,11,12,24,6,25)$.

$3K_{41}$ $V = Z_{41}$. C as follows, cycled modulo 41:

$(0,1,2,3,4,6,8,10)$, $(0,3,6,10,4,9,1,17)$, $(0,7,15,23,9,18,1,32)$, $(0,10,22,11,12,24,1,25)$, $(0,13,30,15,16,32,11,37)$.

$\lambda = 4$

$4K_{9,9}$ $V = \{i_1 \mid 0 \le i \le 8\} \cup \{i_2 \mid 0 \le i \le 8\}$ C as follows, cycled modulo $(8,-)$:

$(0_1,0_2,1_1,1_2,2_2,2_1,3_2,3_1)$, $(0_1,1_2,2_1,3_2,2_2,4_1,6_2,8_1)$, $(0_1,3_2,6_1,5_2,4_2,7_1,2_2,8_1)$.

$4K_{10}$ $V = \{i_j \mid 0 \le i \le 4;\ \ j = 1,2\}$. C as follows, cycled modulo $(5,-)$:

$(0_1,1_1,2_1,3_1,4_1,0_2,1_2,2_2)$, $(0_1,2_1,4_1,0_2,3_1,3_2,1_2,2_2)$, $(0_1,0_2,2_1,2_2,3_2,1_2,4_2,1_1)$.

$\boxed{4K_{19}}$ $V = Z_{19}$. C as follows, cycled modulo 19:

$$(0,1,2,3,4,5,7,9),\quad (0,2,4,7,3,6,10,15),\quad (0,5,13,6,8,14,4,15).$$

$\boxed{4K_{22}}$ $V = \{i_j \mid 0 \le i \le 10;\ j = 1,2\}$. C as follows, cycled modulo $(11,-)$:

$(0_1,1_1,2_1,3_1,4_1,5_1,7_1,9_1)$, $(0_1,2_1,4_1,7_1,3_1,6_1,0_2,1_2)$,
$(0_1,5_1,0_2,1_2,2_2,3_2,1_1,4_2)$, $(0_1,0_2,1_1,1_2,2_2,2_1,3_2,3_1)$,
$(0_1,1_2,2_1,5_2,3_2,6_1,9_2,0_2)$, $(0_1,4_2,8_1,5_2,6_2,2_2,1_2,10_2)$,
$(0_1,4_2,1_2,8_2,6_2,9_2,5_1,2_2)$.

$\boxed{4K_{31}}$ $V = Z_{31}$. C as follows, cycled modulo 31:

$(0,1,2,3,4,5,7,9)$, $(0,2,4,7,3,6,10,15)$, $(0,5,11,17,7,14,1,24)$,
$(0,8,17,9,10,19,1,20)$, $(0,10,22,11,12,25,9,26)$.

$\boxed{\lambda = 6}$

$\boxed{6K_{5,6}}$ $V = \{i_1 \mid 0 \le i \le 4\} \cup \{i_2 \mid 0 \le i \le 5\}$ C as follows, uncycled:

$(0_1,0_2,1_1,1_2,2_2,2_1,3_2,3_1)$, $(0_1,0_2,1_1,1_2,2_2,2_1,3_2,4_1)$,
$(0_1,3_2,3_1,4_2,5_2,4_1,1_2,2_1)$, $(0_1,3_2,1_1,0_2,1_2,2_1,2_2,3_1)$,
$(0_1,3_2,3_1,2_2,4_2,4_1,5_2,1_1)$, $(1_1,1_2,2_1,4_2,5_2,4_1,0_2,3_1)$,
$(1_1,1_2,0_1,0_2,4_2,2_1,5_2,3_1)$, $(1_1,2_2,0_1,3_2,5_2,4_1,4_2,2_1)$,
$(1_1,2_2,2_1,5_2,4_2,4_1,0_2,3_1)$, $(1_1,2_2,0_1,5_2,0_2,3_1,1_2,4_1)$,
$(1_1,3_2,3_1,1_2,4_2,4_1,2_2,2_1)$, $(2_1,0_2,0_1,5_2,3_2,4_1,4_2,3_1)$,
$(2_1,0_2,0_1,3_2,1_2,4_1,2_2,1_1)$, $(2_1,2_2,3_1,4_2,5_2,4_1,1_2,0_1)$,
$(3_1,0_2,1_1,5_2,3_2,4_1,4_2,0_1)$.

$\boxed{6K_{6,7}}$ $V = \{i_1 \mid 0 \leq i \leq 5\} \cup \{i_2 \mid 0 \leq i \leq 6\}$ C as follows, uncycled:

$(0_1, 0_2, 1_1, 1_2, 2_2, 4_1, 5_2, 2_1)$, $(0_1, 3_2, 1_1, 4_2, 5_2, 3_1, 2_2, 5_1)$,
$(0_1, 3_2, 1_1, 0_2, 6_2, 2_1, 1_2, 3_1)$, $(0_1, 4_2, 2_1, 5_2, 6_2, 4_1, 0_2, 5_1)$,
$(1_1, 2_2, 3_1, 4_2, 6_2, 4_1, 3_2, 5_1)$, $(1_1, 2_2, 0_1, 0_2, 1_2, 2_1, 3_2, 3_1)$,
$(1_1, 3_2, 4_1, 5_2, 6_2, 5_1, 0_2, 2_1)$, $(1_1, 4_2, 0_1, 5_2, 6_2, 2_1, 2_2, 3_1)$,
$(1_1, 4_2, 0_1, 0_2, 1_2, 3_1, 2_2, 4_1)$, $(1_1, 5_2, 0_1, 3_2, 6_2, 4_1, 1_2, 5_1)$,
$(2_1, 0_2, 3_1, 3_2, 1_2, 5_1, 5_2, 4_1)$, $(2_1, 2_2, 4_1, 6_2, 5_2, 5_1, 4_2, 3_1)$,
$(2_1, 2_2, 0_1, 3_2, 4_2, 1_1, 0_2, 3_1)$, $(2_1, 4_2, 0_1, 5_2, 6_2, 4_1, 1_2, 5_1)$,
$(3_1, 1_2, 1_1, 5_2, 2_2, 5_1, 3_2, 4_1)$, $(3_1, 1_2, 0_1, 6_2, 0_2, 2_1, 2_2, 5_1)$,
$(3_1, 2_2, 1_1, 6_2, 4_2, 5_1, 1_2, 0_1)$, $(3_1, 3_2, 0_1, 6_2, 5_2, 4_1, 1_2, 2_1)$,
$(4_1, 0_2, 0_1, 6_2, 4_2, 5_1, 5_2, 1_1)$, $(4_1, 0_2, 1_1, 2_2, 4_2, 2_1, 3_2, 5_1)$,
$(4_1, 3_2, 2_1, 4_2, 1_2, 5_1, 0_2, 3_1)$.

$\boxed{6K_{12}}$ $V = Z_{11} \cup \{\infty\}$. C as follows, cycled modulo 11:

$(0, 1, 2, 3, 4, 5, 6, 7)$, $(0, 2, 4, 6, 3, 5, 1, \infty)$, $(0, 2, 7, 5, 3, 8, 1, \infty)$.

$\boxed{6K_{20}}$ $V = Z_{19} \cup \{\infty\}$. C as follows, cycled modulo 19:

$(0, 1, 2, 3, 4, 5, 6, 7)$, $(0, 2, 4, 6, 3, 5, 1, 8)$, $(0, 2, 5, 9, 6, 15, 10, 1)$,
$(0, 5, 11, 6, 7, 13, 1, \infty)$, $(0, 7, 15, 8, 10, 18, 6, \infty)$.

$\boxed{6K_{21}}$ $V = Z_{21}$. C as follows, cycled modulo 21:

$(0, 1, 2, 3, 4, 5, 6, 7)$, $(0, 2, 4, 6, 3, 5, 1, 8)$, $(0, 2, 5, 9, 6, 11, 16, 1)$,
$(0, 5, 11, 6, 7, 13, 1, 14)$, $(0, 7, 16, 8, 10, 17, 5, 19)$.

$\boxed{6K_{29}}$ $V = Z_{29}$. C as follows, cycled modulo 29:

$(0, 1, 2, 3, 4, 5, 6, 7)$, $(0, 2, 4, 6, 3, 5, 1, 8)$, $(0, 2, 5, 9, 6, 11, 16, 1)$,
$(0, 5, 10, 16, 6, 12, 1, 22)$, $(0, 7, 15, 8, 9, 16, 1, 17)$, $(0, 7, 18, 8, 10, 20, 3, 22)$,
$(0, 10, 22, 11, 12, 25, 6, 23)$.

$\boxed{\lambda = 12}$

$12K_{5,5}$ $(5,-)$: $V = \{i_1 \mid 0 \le i \le 4\} \cup \{i_2 \mid 0 \le i \le 4\}$ C as follows, cycled modulo

$(0_1,0_2,1_1,1_2,2_2,2_1,3_2,3_1)$, $(0_1,0_2,1_1,1_2,2_2,2_1,3_2,3_1)$,
$(0_1,0_2,1_1,1_2,2_2,2_1,3_2,3_1)$, $(0_1,1_2,2_1,3_2,2_2,3_1,0_2,1_1)$,
$(0_1,1_2,2_1,3_2,2_2,3_1,0_2,1_1)$.

$12K_{5,7}$ $V = \{i_1 \mid 0 \le i \le 4\} \cup \{i_2 \mid 0 \le i \le 6\}$ C as follows, uncycled:

$(0_1,0_2,4_1,1_2,6_2,3_1,5_2,1_1)$, $(0_1,0_2,4_1,1_2,6_2,3_1,5_2,2_1)$,
$(0_1,2_2,4_1,3_2,5_2,2_1,6_2,1_1)$, $(0_1,2_2,3_1,3_2,4_2,2_1,1_2,1_1)$,
$(0_1,0_2,4_1,1_2,6_2,3_1,4_2,1_1)$, $(0_1,2_2,4_1,4_2,5_2,3_1,3_2,1_1)$,
$(0_1,3_2,4_1,5_2,4_2,3_1,2_2,2_1)$, $(0_1,0_2,4_1,1_2,6_2,2_1,5_2,3_1)$,
$(0_1,2_2,3_1,4_2,3_2,1_1,1_2,2_1)$, $(1_1,0_2,3_1,4_2,6_2,2_1,3_2,4_1)$,
$(0_1,0_2,4_1,1_2,6_2,1_1,5_2,2_1)$, $(0_1,2_2,3_1,5_2,4_2,2_1,1_2,1_1)$,
$(1_1,0_2,3_1,3_2,2_2,2_1,4_2,4_1)$, $(0_1,0_2,4_1,1_2,6_2,3_1,4_2,1_1)$,
$(0_1,2_2,4_1,3_2,5_2,3_1,6_2,2_1)$, $(0_1,3_2,2_1,4_2,5_2,1_1,2_2,4_1)$,
$(0_1,0_2,4_1,6_2,5_2,2_1,4_2,3_1)$, $(0_1,1_2,3_1,3_2,2_2,2_1,0_2,1_1)$,
$(3_1,1_2,2_1,3_2,2_2,1_1,6_2,4_1)$, $(0_1,0_2,4_1,1_2,6_2,2_1,5_2,1_1)$,
$(0_1,2_2,2_1,3_2,5_2,1_1,4_2,3_1)$, $(1_1,0_2,3_1,2_2,3_2,2_1,1_2,4_1)$,
$(0_1,0_2,4_1,1_2,6_2,2_1,5_2,3_1)$, $(0_1,2_2,3_1,4_2,3_2,2_1,1_2,1_1)$,
$(1_1,0_2,3_1,6_2,4_2,2_1,3_2,4_1)$, $(0_1,0_2,4_1,1_2,5_2,1_1,6_2,2_1)$,
$(0_1,2_2,3_1,6_2,4_2,1_1,5_2,4_1)$, $(1_1,0_2,3_1,3_2,2_2,2_1,4_2,4_1)$,
$(0_1,0_2,4_1,1_2,5_2,1_1,6_2,3_1)$, $(0_1,2_2,4_1,5_2,4_2,3_1,3_2,2_1)$,
$(0_1,3_2,2_1,6_2,4_2,1_1,2_2,4_1)$, $(0_1,0_2,3_1,3_2,6_2,1_1,5_2,4_1)$,
$(0_1,1_2,3_1,4_2,3_2,1_1,0_2,2_1)$, $(0_1,2_2,2_1,5_2,4_2,1_1,1_2,4_1)$,
$(1_1,0_2,4_1,2_2,1_2,2_1,6_2,3_1)$.

$12K_{7,7}$ $(7,-)$: $V = \{i_1 \mid 0 \le i \le 6\} \cup \{i_2 \mid 0 \le i \le 6\}$ C as follows, cycled modulo

$(5_1,1_2,4_1,5_2,2_2,6_1,6_2,1_1)$, $(6_2,4_1,1_2,2_1,1_1,2_2,6_1,0_2)$,
$(3_1,6_2,6_1,2_2,4_2,5_1,1_2,2_1)$, $(0_1,0_2,1_1,1_2,2_2,2_1,3_2,3_1)$,
$(0_1,0_2,1_1,1_2,2_2,2_1,3_2,4_1)$, $(0_1,0_2,1_1,3_2,5_2,3_1,6_2,2_1)$,
$(0_1,1_2,4_1,3_2,4_2,5_1,2_2,6_1)$.

$12K_{11}$ $V = Z_{11}$. C as follows, cycled modulo 11:

$(0,1,2,3,4,5,6,7)$, $(0,1,2,3,4,5,6,7)$, $(0,2,4,6,3,5,1,8)$,
$(0,2,4,6,3,5,7,1)$, $(0,2,8,3,5,10,4,9)$.

$\boxed{12K_{14}}$ $V = Z_{13} \cup \{\infty\}$. C as follows, cycled modulo 13:

$(0,1,2,3,4,5,6,7)$, $(0,1,2,3,4,5,6,7)$, $(0,2,4,6,3,5,1,8)$,
$(0,2,4,6,3,5,1,\infty)$, $(0,2,4,6,3,7,12,\infty)$, $(0,3,9,4,5,10,2,\infty)$,
$(0,5,11,6,7,12,4,\infty)$.

$\boxed{12K_{15}}$ $V = Z_{15}$. C as follows, cycled modulo 15:

$(0,1,2,3,4,5,6,7)$, $(0,1,2,3,4,5,6,7)$, $(0,2,4,6,3,5,1,8)$,
$(0,2,4,6,3,5,1,8)$, $(0,2,4,7,3,12,8,13)$, $(0,5,11,6,7,12,3,13)$,
$(0,5,11,6,7,12,4,13)$.

$\boxed{12K_{18}}$ $V = Z_{17} \cup \{\infty\}$. C as follows, cycled modulo 17:

$(0,1,2,3,4,5,6,7)$, $(0,1,2,3,4,5,6,7)$, $(0,2,4,6,3,5,1,8)$,
$(0,2,4,6,3,5,1,8)$, $(0,2,4,7,3,8,15,11)$, $(0,5,11,6,7,12,1,\infty)$,
$(0,5,11,6,7,12,1,\infty)$, $(0,5,13,8,9,14,4,\infty)$, $(0,6,15,9,8,16,7,\infty)$.

$\boxed{12K_{23}}$ $V = Z_{23}$. C as follows, cycled modulo 23:

$(0,1,2,3,4,5,6,7)$, $(0,1,2,3,4,5,6,7)$, $(0,2,4,6,3,5,1,8)$,
$(0,2,4,6,3,5,1,8)$, $(0,2,4,7,3,8,12,16)$, $(0,5,10,15,6,11,2,20)$,
$(0,5,10,15,6,12,4,21)$, $(0,6,13,7,8,15,1,16)$, $(0,7,15,8,9,16,2,18)$,
$(0,7,19,8,10,20,9,21)$, $(0,9,20,10,11,21,8,22)$.

$\boxed{12K_{26}}$ $V = Z_{25} \cup \{\infty\}$. C as follows, cycled modulo 25:

$(0,1,2,3,4,5,6,7)$, $(0,1,2,3,4,5,6,7)$, $(0,2,4,6,3,5,1,8)$,
$(0,2,4,6,3,5,1,8)$, $(0,2,4,7,3,8,12,16)$, $(0,5,10,15,6,11,1,20)$,
$(0,5,10,15,6,11,2,21)$, $(0,6,13,7,8,15,1,16)$, $(0,7,15,8,9,16,1,17)$,
$(0,7,15,8,9,16,1,\infty)$, $(0,8,18,10,9,19,5,\infty)$, $(0,8,21,10,12,22,9,\infty)$,
$(0,11,23,12,13,24,10,\infty)$.

$\boxed{12K_{27}}$ $V = Z_{27}$. C as follows, cycled modulo 27:

$(0,1,2,3,4,5,6,7)$, $(0,1,2,3,4,5,6,7)$, $(0,2,4,6,3,5,1,8)$,
$(0,2,4,6,3,5,1,8)$, $(0,2,4,7,3,8,12,16)$, $(0,5,10,15,6,11,1,20)$,
$(0,5,10,15,6,11,1,21)$, $(0,6,12,18,7,13,5,25)$, $(0,7,15,8,9,16,1,17)$,
$(0,8,17,9,10,18,1,19)$, $(0,8,18,10,11,19,4,21)$, $(0,10,21,11,12,23,8,24)$,
$(0,11,23,12,13,24,10,25)$.

$\boxed{12K_{30}}$ $V = Z_{29} \cup \{\infty\}$. C as follows, cycled modulo 29:

$(0,1,2,3,4,5,6,7)$, $(0,1,2,3,4,5,6,7)$, $(0,2,4,6,3,5,1,8)$,
$(0,2,4,6,3,5,1,8)$, $(0,2,4,7,3,8,12,16)$, $(0,5,10,15,6,11,1,20)$,
$(0,5,10,15,6,11,1,21)$, $(0,6,12,18,7,13,3,25)$, $(0,7,14,21,8,16,6,28)$,
$(0,8,17,9,10,18,1,19)$, $(0,8,17,9,10,18,1,20)$, $(0,8,22,11,12,23,5,\infty)$,
$(0,11,23,12,13,24,6,\infty)$, $(0,12,25,13,14,26,9,\infty)$, $(0,12,26,13,14,27,11,\infty)$.

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