2-WALKS IN 3-CONNECTED PLANAR GRAPHS

ZHICHENG GAO
R. BRUCE RICHTER

Department of Mathematics and Statistics
Carleton University
Ottawa, CANADA K1S 5B6

XINGXING YU

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332, U.S.A.

ABSTRACT. In this article, we prove that every 3-connected planar graph has a closed walk visiting each vertex, none more than twice, such that any vertex visited twice is in a vertex cut of size 3. This generalizes both Tutte's Theorem that 4-connected planar graphs are Hamiltonian and the result of Gao and Richter that 3-connected planar graphs have a closed walk visiting each vertex at least once but at most twice.

1. INTRODUCTION

Tutte [**Tu**] proved that every 4-connected planar graph is Hamiltonian. Recently, Gao and Richter [**GR**] settled a conjecture of Jackson and Wormald [**JW**] by showing that every 3-connected planar graph has a *closed 2-walk* — a closed walk that visits every vertex at least once but at most twice. In this paper we prove a common refinement of these results, which was conjectured by Thomas [**T**]. A *k-cut* in G is a set A of vertices such that $G - A$ is not connected and $|A| = k$.

Theorem 1. *Let G be a 3-connected planar graph and let x, y be two vertices both incident with the same face of G. Then there is a closed 2-walk W in G visiting x and y only once each, such that every vertex visited twice by W is in a 3-cut in G.*

That Theorem 1 generalizes Tutte's Theorem is obvious: if G is 4-connected and W is the closed 2-walk guaranteed by Theorem 1, then W must be a Hamilton cycle, since G has no 3-cuts and, therefore, W can have no repeated vertices.

The same ideas improve Thomassen's Theorem [**Th**] that 4-connected planar graphs are Hamilton-connected. A *2-walk* is a walk visiting each vertex at least once but at most twice.

Theorem 2. *Let G be a 3-connected planar graph and let x and y be any vertices of G. Then there is a 2-walk W in G from x to y such that any vertex visited twice by W is in a 3-cut of G.*

We remark that the proofs given in this paper are substantially simpler than those of [**GR**]. However, their proofs form the core for the results by Brunet et al [**BEGMR**], where it is proved that every 3-connected graph that embeds in either the torus or the Klein bottle has a 2-walk. It would be of substantial interest to know if Theorems 1 and 2 generalize to these graphs.

2. Circuit Graphs

We shall in fact prove our results for circuit graphs, a class of planar graphs that includes the 3-connected planar graphs.

A *circuit graph* is an ordered pair (G, C) consisting of a 2-connected planar graph G and a cycle C of G such that, in some embedding of G in the plane, C bounds a face and, for every 2-cut A in G, every component of $G - A$ contains a vertex of C.

Obviously, if C is a face boundary of a 3-connected planar graph G, then (G, C) is a circuit graph. Circuit graphs have some very nice inductive properties. The ones relevant for this work are stated in the following result. Proofs can be found in [**GR**]. A *plane chain of blocks* is a graph, embedded in the plane, with blocks $B_1, B_2, \ldots, B_k$, such that, for each $i = 2, 3, \ldots, k$, B_{i-1} and B_i have a vertex in common, no two of which are the same, and, for each $j = 1, 2, \ldots, k$, $\bigcup_{i \neq j} B_i$ is in the infinite face of B_j.

Lemma 3. *Let (G, C) be a circuit graph.*

(1) *Let G be embedded in the plane with C bounding the infinite face and let C' be any cycle of G. Let H be the subgraph of G contained in the closed disc bounded by C'. Then (H, C') is a circuit graph.*

(2) *If $v \in V(C)$, then $G - v$ is a plane chain of blocks $B_1, \ldots, B_k$. Moreover, one of the neighbours of v in C is in B_1 and the other is in B_k.*

3. Tutte Paths and Tutte Cycles

In order to prove Theorem 1, we shall first prove the existence of a "Tutte path" and a "Tutte cycle" in a circuit graph. For a subgraph J of a graph G, a *J-bridge* in G is a component K of $G - V(J)$, together with the edges of G joining a vertex of K to a vertex of J and the ends of such edges. If L is a J-bridge, then the vertices in $V(L) \cap V(J)$ are the *vertices of attachment* of L.

We remark that the usual definition of J-bridge allows the possibility of an edge, not in J, together with its ends, which are in J. Such bridges are of no concern to us, and, to simplify the later discussion, we have chosen not to include them in the definition used in this article.

A *Tutte path* (*Tutte cycle*) in a circuit graph (G, C) is a path (cycle) P such that every P-bridge has at most 3 vertices of attachment and any P-bridge containing an edge of C has at most 2 vertices of attachment.

We abbreviate system of distinct representatives to SDR. If J is a subgraph of a graph G, then a *SDR of the J-bridges* is a SDR of the sets $\{V(L) \cap V(J) \mid L \text{ is a } J\text{-bridge}\}$.

Theorem 4. *Let (G, C) be a circuit graph and let $x, u \in V(C)$, let $y \in V(G)$ with $x \neq y$ and let $a \in \{x, u\}$. Then there is a Tutte path P in G from x to y through u and a SDR S of the P-bridges such that $a \notin S$.*

Proof. The proof proceeds by induction on $|E(G)|$. The unique smallest circuit graph is K_3, for which the result is trivial. For the inductive step, we may suppose G is embedded in the plane so that C is the boundary of the infinite face.

If $u = x$, then pick any other vertex of $V(C - x)$ and let it be u. (Of course we do not change $a = x$.) Thus, we can assume that $u \notin \{x, y\}$. The case $u = y$ and $a \neq u$ can be similarly dismissed, while if $a = u = y$, then interchange the roles of x and y and proceed as above.

For any two distinct vertices r, s of C, let rCs denote the clockwise path in C from r to s. Thus, the two paths in C between x and u are xCu and uCx. We can assume that the drawing is such that y is not in xCu and that uCx has length at least 2. Let u_1 be the neighbour of u in the path uCx. It is possible that $u_1 = y$, in which case we let $K = \{u_1\}$, $\hat{P} = \{u_1\}$ and $\hat{S} = \emptyset$.

If $u_1 \neq y$, then let K be the minimal connected union of blocks of $G - xCu$ containing both u_1 and y. (Throughout this work, if H is a subgraph of a graph G, then $G - H$ denotes the subgraph $G - V(H)$ of G.) Clearly, K is a plane chain of blocks $B_1, B_2, \ldots, B_\ell$, with $u_1 \in V(B_1)$ and $y \in V(B_\ell)$. For $i = 1, 2, \ldots, \ell - 1$, let v_i be the vertex common to B_i and B_{i+1}. Set $v_0 = u_1$ and $v_\ell = y$.

If $B_1 \cap C$ is not just u_1, then let k be the largest index such that B_k contains an edge of C. Otherwise, set $k = 1$. Let w be the vertex in B_k nearest x in uCx.

For $1 \leq i \leq \ell$, either B_i is just $v_{i-1}v_i$ and its ends or (B_i, C_i) is a circuit graph, where C_i bounds the infinite face of B_i. In the first case, let $P_i = (v_{i-1}, v_{i-1}v_i, v_i)$ and $S_i = \emptyset$.

For $i \in \{1, 2, \ldots, \ell\} \setminus \{k\}$, the inductive assumption yields a Tutte path P_i in B_i from v_{i-1} to v_i and a SDR S_i of the P_i-bridges in B_i such that either $v_i \notin S_i$ (if $i < k$) or $v_{i-1} \notin S_i$ (if $i > k$).

Indutively there is a Tutte path P_k in B_k from v_{k-1} to v_k through w and a SDR of the P_k-bridges in B_k such that $w \notin S_k$.

Let $\hat{P} = \bigcup_{i=1}^{\ell} P_i$ and $\hat{S} = \bigcup_{i=1}^{\ell} S_i$. Set $\hat{K} = K \cup xCu_1$.

We now extend $\hat{P}$ back to x. For each $\hat{K}$-bridge L in G, $L \cap K$ consists of at most one vertex, which we call $a(L)$. Let $\hat{L}$ be the bridge (if there is one) containing the path wCx. Because (G, C) is a circuit graph, this is the only $\hat{K}$-bridge in G that can have only two vertices of attachment. If $\hat{L}$ has only two vertices of attachment, then we shall do nothing with it; w will be its representative.

Let F' denote the union of xCu, all $\hat{K}$-bridges in G and all $\hat{P}$-bridges in K that contain a vertex $a(L)$ that is not in $\hat{P}$. Let $F = F' - \hat{P}$. Let $a_1, a_2, \ldots, a_s$ be the cut vertices of F that are in xCu, in the order they appear from x to u. Note that $a_1, \ldots, a_s$ do not include x and u, i.e. they are internal to the path xCu. Let $a_0 = x$ and $a_{s+1} = u$.

Either there is a path in F from a_{i-1} to a_i that is disjoint from $a_{i-1}Ca_i$ (except for their common ends) or there is not. If there is not, then a_{i-1} and a_i are consecutive vertices of xCu and we set Q_i to be the path $(a_{i-1}, a_{i-1}a_i, a_i)$ and $R_i = \emptyset$.

Otherwise, let A'_i be the block of F containing $a_{i-1}Ca_i$ and let A_i be the union of A'_i and any A'_i-bridge in F that does not contain either a_{i-1} or a_i. There is a $\hat{K}$-bridge L_i that has an edge in A'_i. If there is no vertex $a(L_i)$, then clearly $A_i = A'_i$. If there is a vertex $a(L_i)$ and it is not in $\hat{P}$, then clearly $A_i = A'_i \cup (M_i - \hat{P})$, for some $\hat{P}$-bridge M_i in K. Finally, if if $a(L_i)$ is in $\hat{P}$, then, because (G, C) is a circuit graph, for each vertex p of L_i, there are three disjoint paths from p to the vertices $a_{i-1}, a_i, a(L_i)$. Therefore, $L_i - a(L_i)$ is 2-connected. It follows that $A_i = A'_i$.

Let C'_i be the cycle bounding the infinite face of A'_i, so that (A'_i, C'_i) is a circuit graph.

If $\hat{L}$ has at least 3 vertices of attachment, then $\hat{L} - w \subseteq A_1$. Let z be the vertex of $A'_1 \cap C$ such that $zCx = A'_1 \cap C$. (It is possible that $z = x$, in which case zCx is also just x.) Inductively, there is a Tutte path Q_1 in A'_1 from x to a_1 through z and a SDR R_1 of the Q_1-bridges of A'_1 such that either $x \notin R_1$ (if $a = x$) or $a_1 \notin R_1$ (if $a = u$).

Now we treat the remaining A'_i, $i = 1, 2, \ldots, s+1$; we need to deal with the case $i = 1$ only if $\hat{L}$ has only two vertices of attachment. We remind the reader that we are assuming that (A'_i, C'_i) is a circuit graph, as otherwise we have already obtained the path Q_i and the SDR R_i.

If $A_i \cap K$ is not empty, then $A_i = A'_i \cup (M_i - \hat{P})$. Let z be the vertex in $A'_i \cap M_i$. If $A_i \cap K$ is empty, then let z be any vertex in C'_i. Inductively, there is a Tutte path Q_i in A'_i from a_{i-1} to a_i through z and a SDR R_i of the Q_i-bridges such that either $a_{i-1} \notin R_i$ (if $a = x$) or $a_i \notin R_i$ (if $a = u$).

The required Tutte path in G is $P = (\bigcup_{i=1}^{s+1} Q_i) \cup (u, uu_1, u_1) \cup \hat{P}$ with $S = (\bigcup_{i=1}^{s+1} R_i) \cup \hat{S} \cup \{w\}$ as the required SDR of the P-bridges in G.□

The following consequence of Theorem 4 is the heart of the proof of Theorem 1.

Corollary 5. *Let (G, C) be a circuit graph and let $x, y \in V(C)$. Then there is a Tutte cycle T in G and a SDR S of the T-bridges in G with $x, y \in V(T)$ and $x, y \notin S$.*

Proof. Let x have neighbours u and v in C. The graph $G - x$ is a plane chain of blocks $B_1, B_2, \ldots, B_k$, with $u \in V(B_1)$ and $v \in V(B_k)$. Let j be least such that $y \in V(B_j)$. For $i = 1, 2, \ldots, k-1$, let v_i be the vertex common to B_i and B_{i+1}, let $v_0 = u$ and $v_k = v$.

For $i = 1, 2, \ldots, k$, if B_i is just the edge $v_{i-1}v_i$ and its ends, then we set $P_i = (v_{i-1}, v_{i-1}v_i, v_i)$ and $S_i = \emptyset$.

Otherwise, for $1 \leq i < j$, by Theorem 4 there is a Tutte path P_i from v_{i-1} to v_i in B_i having a SDR S_i of the P_i-bridges in B_i, such that $v_i \notin S_i$. Let P_j be a Tutte path in B_j from v_{j-1} to v_j through y in B_j having a SDR S_j of the P_j-bridges in B_j, such that $y \notin S_j$. For $j < i \leq k$, let P_i be a Tutte path in B_i from v_{i-1} to v_i having a SDR S_i of the P_i-bridges in B_i, such that $v_{i-1} \notin S_i$.

The cycle obtained by adding x, xu and xv to the path $P_1 \cup P_2 \cup \cdots \cup P_k$ is the desired Tutte cycle and $S = \bigcup_{i=1}^{k} S_i$ is the required SDR.□

4. Proof of Theorems 1 and 2

In this section we use Theorem 4 to prove Theorems 1 and 2 for circuit graphs. If (G, C) is a circuit graph, an *internal k-cut* of G is a k-cut A of G such that $G - A$ contains a component disjoint from C.

Theorem 6. *Let (G, C) be a circuit graph and let $x, y \in V(C)$. Then there is a closed 2-walk W in G visiting x and y only once each such that any vertex visited twice by W is in either a 2-cut or an internal 3-cut of G.*

Proof. In fact, we shall prove something slightly stronger. We shall require that if v is a vertex of G visited twice by W, then either v is in an internal 3-cut or there is a 2-cut $\{v, w\}$ of G with v and w both in the same path in C from x to y, i.e. either both are in xCy or both are in yCx.

We proceed by induction on $|E(G)|$, with the case $|E(G)| = 3$ being trivial. For the inductive step, we can suppose that G is drawn in the plane so that C bounds the infinite face.

By Corollary 5, G has a Tutte cycle T through x and y and a SDR S for the T-bridges of G with $x, y \notin S$. We use this to construct the desired closed 2-walk.

Let L be a T-bridge and let s be the representative of L in S. If L has only two vertices of attachment, then L contains an edge of C (as otherwise (G, C) is not a circuit graph). The only other possibility is that L has exactly 3 vertices of attachment.

Suppose first that L has exactly two vertices of attachment, say s and s'. Let sCs' denote the path $C \cap L$ and let t be the neighbour of s' in sCs'. By Lemma 3 (2), $L - s'$ is a plane chain of blocks $B_1, B_2, \ldots, B_m$, with $s \in V(B_1)$, $s \notin V(B_2)$, $t \in V(B_m)$ and $t \notin V(B_{m-1})$. For $i = 1, 2, \ldots, m-1$, let v_i be the vertex common to B_i and B_{i+1} and let $v_0 = s$, $v_m = t$.

For $i = 1, 2, \ldots, m$, either B_i is just the edge $v_{i-1}v_i$ and its ends or C_i is a cycle bounding the infinite face of B_i and (B_i, C_i) is a circuit graph. Moreover, $C_i \cap C$ is a path.

In the first case, we let $W_i = (v_{i-1}, v_{i-1}v_i, v_i, v_iv_{i-1}, v_{i-1})$. In the second case, inductively, there is a closed 2-walk W_i in B_i visiting each of v_{i-1} and v_i only once such that any vertex visited twice by W is in either a 2-cut of B_i or an internal 3-cut of B_i.

Now suppose L has three vertices of attachment, say s, s' and s''. Then L is disjoint from C except possibly for vertices of attachment. We claim that $L - \{s', s''\}$ is a plane chain of blocks $B_1, B_2, \ldots, B_m$, with $s \in B_1$, $s \notin B_2$. We can add the edges $ss', ss'', s's''$ to L to create a circuit graph (L', C'), where C' is the triangle through the new edges and L' is L with the three new edges. Deleting s' yields, by Lemma 3 (2), a plane chain of blocks with s in one leaf block and s'' in the other. Since they are adjacent, there is only one block. Therefore, $L' - s'$ is 2-connected and now Lemma 3 (2) shows that $L - \{s', s''\} = L' - \{s', s''\}$ is a plane chain of blocks, as required.

There is a vertex v_m in B_m that is in a face boundary of G with both s' and s''. We proceed exactly as in the case L has only two vertices of attachment.

It is important to observe that any internal 3-cut of B_i is an internal 3-cut of G and the 2-cuts of B_i that we need to consider (i.e. both vertices in either $v_{i-1}C_iv_i$ or both vertices in $v_iC_iv_{i-1}$) are either 2-cuts of G or are contained in internal 3-cuts of G. It is clear that we can get a closed 2-walk in G by traversing T from one representative to the next and then detouring into the bridges using the walks W_i, being careful to go from v_{i-1} to v_i on W_i, and then going into B_{i+1} before returning from v_i to v_{i-1} on the remainder of W_i.□

The appropriate generalization of Theorem 2 to circuit graphs is the following. It follows from Theorem 4 in the same way that Theorem 6 follows from Corollary 5.

Theorem 7. *Let (G, C) be a circuit graph let $x \in V(C)$ and let $y \in V(G)$. Then there is a 2-walk from x to y in G such that any vertex visited twice by W is in either a 2-cut or an internal 3-cut of G.*

REFERENCES

[B] D.W. Barnette, *Trees in polyhedral graphs*, Canad. J. Math. **18** (1966), 731-736.

[BEGMR] R. Brunet, M.N. Ellingham, Z. Gao, A. Metzlar and R.B. Richter, *Spanning planar subgraphs of graphs in the torus and Klein bottle*, submitted.

[GR] Z. Gao and R.B. Richter, *2-walks in circuit graphs*, to appear, J. Combin. Theory, Ser. B.

[JW] B. Jackson and N.C. Wormald, *k-walks of graphs*, Australasian J. of Combin. **2** (1990), 135-146.

[T] R. Thomas, personal communication.

[Th] C. Thomassen, *A theorem on paths in planar graphs*, J. Graph Theory **7** (198), 169-176.

[Tu] W.T. Tutte, *A theorem on planar graphs*, Trans. Amer. Math. Soc. **82** (1956), 99-116.

(Received 16/2/94)