Classroom Note

Almost-Isoceles Right-Angled Triangles

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Abstract. We provide an elementary method to show that there exist infinitely many right-angled triangles with integral sides in which the lengths of the two non-hypotenuse sides differ by 1. The method also enables us to construct all such right-angled triangles recursively.

1. Introduction

There does not exist any isoceles right-angled triangle with integral sides. Does there exist a right-angled triangle with integral sides in which the lengths of the two non-hypotenuse sides differ by 1? We shall call such a triangle an almost-isoceles right-angled (AIRA) triangle. For an AIRA-triangle, there exist positive integers $x$ and $y$ such that the lengths of the sides are $x, x+1$ and $y$ respectively with $x^2+(x+1)^2 = y^2$. We shall call the triple $(x, x+1, y)$ an AIRA-triple. An immediate example is the triple $(3, 4, 5)$ and another one is $(20, 21, 29)$. We would need a calculator to get the next few: $(119, 120, 169), (696, 697, 985), (4059, 4060, 5741)$, etc.. Are there infinitely many AIRA-triples? If so, is there a way to find all such triples? The answer to both questions is "yes", and one can reduce the problem to a Pell's equation (see [1], p.357) and show that there are infinitely many AIRA-triples. In this note, we shall however use an elementary method to show that there are infinitely many AIRA-triples and that all such triples can be obtained recursively.
2. A Recursive Construction

Consider an AIRA-triple \((x, x + 1, y)\). Thus

\[ x^2 + (x + 1)^2 = y^2. \]

Clearly, the problem of finding all AIRA-triangles is equivalent to finding all positive integer solutions to the following Diophantine equation:

\[ 2x^2 + 2x + 1 = y^2. \]

To solve this, we first write

\[ 4x^2 + 4x + 2 = 2y^2, \]

from which we get

\[ (2x + 1)^2 = 2y^2 - 1. \]

Hence \(2y^2 - 1\) must be a perfect square. There exists a positive integer \(k\) with \(2y^2 - 1 = (y + k)^2\). Then

\[ y^2 - 2ky - (1 + k^2) = 0 \]
\[ \Rightarrow \ (y - k)^2 = 2k^2 + 1. \]

Again, as \(2k^2 + 1\) must also be a perfect square, there exists another positive integer \(t\) with \(2k^2 + 1 = (k + t)^2\). Then

\[ k^2 - 2tk - (t^2 - 1) = 0 \]
\[ \Rightarrow \ (k - t)^2 = 2t^2 - 1. \]

The above derivations suggest the following simultaneous recurrence relations:

\[ (a_n - b_{n-1})^2 = 2b_{n-1}^2 + 1 \quad \text{(I)} \]
\[ (b_n - a_n)^2 = 2a_n^2 - 1 \quad \text{(II)} \]

From (II), we have

\[ (b_{n-1} - a_{n-1})^2 = 2a_{n-1}^2 - 1 \]
\[ \Rightarrow \ b_{n-1}^2 - 2a_{n-1}b_{n-1} = a_{n-1}^2 - 1 \]
\[ \Rightarrow \ 2b_{n-1}^2 + 1 = (b_{n-1} + a_{n-1})^2, \quad \text{and so by (I)} \]
\[ a_n = 2b_{n-1} + a_{n-1}. \quad \text{(III)} \]
In like manner, from (I), we have

\[(a_n - b_{n-1})^2 = 2b_{n-1}^2 + 1\]
\[\Rightarrow a_n^2 - 2a_nb_{n-1} = b_{n-1}^2 + 1\]
\[\Rightarrow 2a_n^2 - 1 = (a_n + b_{n-1})^2, \quad \text{and so by (II)}\]
\[b_n = 2a_n + b_{n-1}. \]  

(IV)

Starting with the initial condition \(a_0 = 1\) and \(b_0 = 2\), the two recurrence relations (III) and (IV) will easily generate infinitely many solutions \((a_n, b_n), n = 0, 1, 2, \ldots\). We shall show via Claims 1–3 below that each \(a_n\) will be the length of the hypotenuse of an AIRA-triangle and conversely, the length of the hypotenuse of any AIRA-triangle is equal to \(a_n\) for some \(n\). In fact, for each \(n = 0, 1, 2, \ldots\), the lengths of the sides of the corresponding AIRA-triangle are \(x_n, x_n + 1\) and \(a_{n+1}\), where \((2x_n + 1)^2 = 2a_{n+1}^2 - 1\).

**Claim 1.** \(a_n, b_n\) and \(x_n\) are positive integers, for each \(n = 0, 1, 2, \ldots\).

**Proof.** With the initial conditions \(a_0 = 1, b_0 = 2, a_n\) and \(b_n\) are clearly positive integers. Also, as \(2a_{n+1}^2 - 1\) is an odd perfect square, \(x_n\) is also an integer. ■

**Claim 2.** \((x_n, x_n + 1, a_{n+1})\) is an AIRA-triple, for each \(n = 0, 1, 2, \ldots\).

**Proof.** We have

\[(2x_n + 1)^2 = 2a_{n+1}^2 - 1\]
\[\Rightarrow 4x_n^2 + 4x_n + 2 = 2a_{n+1}^2\]
\[\Rightarrow 2x_n^2 + 2x_n + 1 = a_{n+1}^2\]
\[\Rightarrow x_n^2 + (x_n + 1)^2 = a_{n+1}^2. \] ■

**Claim 3.** Every AIRA-triple is equal to \((x_n, x_n + 1, a_{n+1})\), for some \(n = 0, 1, 2, \ldots\).

**Proof.** Suppose to the contrary that the claim is not valid. Let \((x, x + 1, y)\) be the AIRA-triple with the smallest \(y\) which is not equal to any of the \((x_n, x_n + 1, a_{n+1})\)'s. Then

\[x^2 + (x + 1)^2 = y^2,\]

from which we get

\[(2x + 1)^2 = 2y^2 - 1.\]

Hence \(2y^2 - 1\) is a perfect square. There exists a positive integer \(b\) with

\[(b + y)^2 = 2y^2 - 1.\]
Then
\[(y - b)^2 = 2b^2 + 1,\]
which implies that \(2b^2 + 1\) is a perfect square. There exists a positive integer \(z\) with
\[(z + b)^2 = 2b^2 + 1.\]
Then \(z < y\) and \((b - z)^2 = 2z^2 - 1\) and so \(2z^2 - 1\) is an odd perfect square. Thus there exists a positive integer \(t\) with \((2t + 1)^2 = 2z^2 - 1\), which implies that \((t, t + 1, z)\) is a AIRA-triple and so by the minimality of \((x, x + 1, y)\), there exists a positive integer \(n\) such that
\[(x_n, x_n + 1, a_{n+1}) = (t, t + 1, z).\]
But then we have
\[(b - z)^2 = 2z^2 - 1\]
\[\Rightarrow (b - a_{n+1})^2 = 2a_{n+1}^2 - 1\]
\[\Rightarrow b = b_{n+1}, \quad \text{by (II)}\]
\[\Rightarrow (y - b_{n+1})^2 = 2b_{n+1}^2 + 1\]
\[\Rightarrow y = a_{n+2}, \quad \text{by (I)},\]
so that
\[(x_{n+1}, x_{n+1} + 1, a_{n+2}) = (x, x + 1, y),\]a contradiction. \[\blacksquare\]

3. **Numerical Computation**

From the argument given in the previous section, we see that, starting from \(a_0 = 1\) and \(b_0 = 2\), we may apply (III), (IV) successively to obtain all the AIRA-triples. We present in the following table the first seven of these triples.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(a_n)</th>
<th>(b_n)</th>
<th>(x_n)</th>
<th>AIRA-triple</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>(3, 4, 5)</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>12</td>
<td>20</td>
<td>(20, 21, 29)</td>
</tr>
<tr>
<td>2</td>
<td>29</td>
<td>70</td>
<td>119</td>
<td>(119, 120, 169)</td>
</tr>
<tr>
<td>3</td>
<td>169</td>
<td>408</td>
<td>696</td>
<td>(696, 697, 985)</td>
</tr>
<tr>
<td>4</td>
<td>985</td>
<td>2378</td>
<td>4059</td>
<td>(4059, 4060, 5741)</td>
</tr>
<tr>
<td>5</td>
<td>5741</td>
<td>13860</td>
<td>23660</td>
<td>(23660, 23661, 33461)</td>
</tr>
<tr>
<td>6</td>
<td>33461</td>
<td>80782</td>
<td>137903</td>
<td>(137903, 137904, 195025)</td>
</tr>
</tbody>
</table>
To end the paper, we would like to point out that the two sequences $a_n$, and $b_n$ actually give all the solutions to the following two Pell's equations.

\[
x^2 - 2y^2 = 1, \quad \text{and} \quad x^2 - 2y^2 = -1,
\]

with $y = b_n$ and $a_n$ respectively.

Reference


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