Classroom Note

Almost-Isoceles Right-Angled Triangles

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Abstract. We provide an elementary method to show that there exist infinitely many right-angled triangles with integral sides in which the lengths of the two non-hypotenuse sides differ by 1. The method also enables us to construct all such right-angled triangles recursively.

1. Introduction

There does not exist any isoceles right-angled triangle with integral sides. Does there exist a right-angled triangle with integral sides in which the lengths of the two non-hypotenuse sides differ by 1? We shall call such a triangle an *almost-isoceles right-angled* (AIRA) triangle. For an AIRA-triangle, there exist positive integers x and y such that the lengths of the sides are $x, x+1$ and y respectively with $x^2+(x+1)^2 = y^2$. We shall call the triple $(x, x+1, y)$ an *AIRA-triple.* An immediate example is the triple $(3,4,5)$ and another one is $(20,21,29)$. We would need a calculator to get the next few: $(119,120,169)$, $(696,697,985)$, $(4059,4060,5741)$, etc.. Are there infinitely many AIRA-triples? If so, is there a way to find all such triples? The answer to both questions is "yes", and one can reduce the problem to a Pell's equation (see [1], p.357) and show that there are infintely many AIRA-triples. In this note, we shall however use an elementary method to show that there are infinitely many AIRA-triples and that all such triples can be obtained recursively.

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2. A Recursive Construction

Consider an AIRA-triple $(x, x+1, y)$. Thus

$$x^2 + (x+1)^2 = y^2.$$

Clearly, the problem of finding all AIRA-triangles is equivalent to finding all positive integer solutions to the following Diophantine equation:

$$2x^2 + 2x + 1 = y^2.$$

To solve this, we first write

$$4x^2 + 4x + 2 = 2y^2,$$

from which we get

$$(2x+1)^2 = 2y^2 - 1.$$

Hence $2y^2 - 1$ must be a perfect square. There exists a positive integer k with $2y^2 - 1 = (y+k)^2$. Then

$$\begin{aligned} & y^2 - 2ky - (1+k^2) = 0 \\ \Rightarrow\quad & (y-k)^2 = 2k^2 + 1. \end{aligned}$$

Again, as $2k^2+1$ must also be a perfect square, there exists another positive integer t with $2k^2 + 1 = (k+t)^2$. Then

$$\begin{aligned} & k^2 - 2tk - (t^2 - 1) = 0 \\ \Rightarrow\quad & (k-t)^2 = 2t^2 - 1. \end{aligned}$$

The above derivations suggest the following simultaneous recurrence relations:

$$(a_n - b_{n-1})^2 = 2b_{n-1}^2 + 1, \tag{I}$$

$$(b_n - a_n)^2 = 2a_n^2 - 1. \tag{II}$$

From (II), we have

$$\begin{aligned} & (b_{n-1} - a_{n-1})^2 = 2a_{n-1}^2 - 1 \\ \Rightarrow\quad & b_{n-1}^2 - 2a_{n-1}b_{n-1} = a_{n-1}^2 - 1 \\ \Rightarrow\quad & 2b_{n-1}^2 + 1 = (b_{n-1} + a_{n-1})^2, \quad \text{and so by (I)} \end{aligned}$$

$$a_n = 2b_{n-1} + a_{n-1}. \tag{III}$$

In like manner, from (I), we have

$$\begin{aligned}
&(a_n - b_{n-1})^2 = 2b_{n-1}^2 + 1 \\
\Rightarrow\quad & a_n^2 - 2a_n b_{n-1} = b_{n-1}^2 + 1 \\
\Rightarrow\quad & 2a_n^2 - 1 = (a_n + b_{n-1})^2, \quad \text{and so by (II)} \\
& b_n = 2a_n + b_{n-1}. \qquad \text{(IV)}
\end{aligned}$$

Starting with the initial condition $a_0 = 1$ and $b_0 = 2$, the two recurrence relations (III) and (IV) will easily generate infinitely many solutions (a_n, b_n), $n = 0, 1, 2, \ldots$. We shall show via Claims 1–3 below that each a_n will be the length of the hypotenuse of an AIRA-triangle and conversely, the length of the hypotenuse of any AIRA-triangle is equal to a_n for some n. In fact, for each $n = 0, 1, 2, \ldots$, the lengths of the sides of the corresponding AIRA-triangle are $x_n, x_n + 1$ and a_{n+1}, where $(2x_n + 1)^2 = 2a_{n+1}^2 - 1$.

Claim 1. a_n, b_n and x_n are positive integers, for each $n = 0, 1, 2, \ldots$.

Proof. With the initial conditions $a_0 = 1$, $b_0 = 2$, a_n and b_n are clearly positive integers. Also, as $2a_{n+1}^2 - 1$ is an odd perfect square, x_n is also an integer. ∎

Claim 2. $(x_n, x_n + 1, a_{n+1})$ is an AIRA-triple, for each $n = 0, 1, 2, \ldots,$.

Proof. We have

$$\begin{aligned}
&(2x_n + 1)^2 = 2a_{n+1}^2 - 1 \\
\Rightarrow\quad & 4x_n^2 + 4x_n + 2 = 2a_{n+1}^2 \\
\Rightarrow\quad & 2x_n^2 + 2x_n + 1 = 2a_{n+1}^2 \\
\Rightarrow\quad & x_n^2 + (x_n + 1)^2 = a_{n+1}^2. \quad ∎
\end{aligned}$$

Claim 3. Every AIRA-triple is equal to $(x_n, x_n + 1, a_{n+1})$, for some $n = 0, 1, 2, \ldots$.

Proof. Suppose to the contrary that the claim is not valid. Let $(x, x + 1, y)$ be the AIRA-triple with the smallest y which is not equal to any of the $(x_n, x_n + 1, a_{n+1})$'s. Then

$$x^2 + (x + 1)^2 = y^2,$$

from which we get

$$(2x + 1)^2 = 2y^2 - 1.$$

Hence $2y^2 - 1$ is a perfect square. There exists a positive integer b with

$$(b + y)^2 = 2y^2 - 1.$$

Then

$$(y-b)^2 = 2b^2 + 1,$$

which implies that $2b^2 + 1$ is a perfect square. There exists a positive integer z with

$$(z+b)^2 = 2b^2 + 1.$$

Then $z < y$ and $(b-z)^2 = 2z^2 - 1$ and so $2z^2 - 1$ is an odd perfect square. Thus there exists a positive integer t with $(2t+1)^2 = 2z^2 - 1$, which implies that $(t, t+1, z)$ is a AIRA-triple and so by the minimality of $(x, x+1, y)$, there exists a positive integer n such that

$$(x_n, x_n + 1, a_{n+1}) = (t, t+1, z).$$

But then we have

$$\begin{aligned} & (b-z)^2 = 2z^2 - 1 \\ \Rightarrow\ & (b-a_{n+1})^2 = 2a_{n+1}^2 - 1 \\ \Rightarrow\ & b = b_{n+1}, \quad \text{by (II)} \\ \Rightarrow\ & (y-b_{n+1})^2 = 2b_{n+1}^2 + 1 \\ \Rightarrow\ & y = a_{n+2}, \quad \text{by (I)}, \end{aligned}$$

so that

$$(x_{n+1}, x_{n+1} + 1, a_{n+2}) = (x, x+1, y),$$

a contradiction. ∎

3. Numerical Computation

From the argument given in the previous section, we see that, starting from $a_0 = 1$ and $b_0 = 2$, we may apply (III), (IV) successively to obtain all the AIRA-triples. We present in the following table the first seven of these triples.

n	a_n	b_n	x_n	AIRA-triple
0	1	2	3	(3, 4, 5)
1	5	12	20	(20, 21, 29)
2	29	70	119	(119, 120, 169)
3	169	408	696	(696, 697, 985)
4	985	2378	4059	(4059, 4060, 5741)
5	5741	13860	23660	(23660, 23661, 33461)
6	33461	80782	137903	(137903, 137904, 195025)

To end the paper, we would like to point out that the two sequences a_n, and b_n actually give all the solutions to the following two Pell's equations.

$$x^2 - 2y^2 = 1, \quad \text{and}$$
$$x^2 - 2y^2 = -1,$$

with $y = b_n$ and a_n respectively.

Reference

[1] Ivan Niven, Herbert S. Zukerman and Hugh L. Montgomery, *An Introduction to the Theory of Numbers,* (Fifth Edition), Wiley, New York, 1991.

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