Classroom Note

Some partitions of S(2,3,v^2) and S(2,4,v^2)

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Dedicated to the memory of Alan Rahilly, 1947 – 1992

Introduction

The aim of this note is to provide a Steiner triple system STS(v^2) which admits a partition into v STS(v) for every $v \equiv 1$ or 3 (mod 6). The STS(v^2) obtained in this way will also contain $\frac{1}{6}v(v-1)$ subsystems of order $3v$. An analogous construction is provided of an $S(2,4,v^2)$ with a partition into v $S(2,4,v)$'s in the case $v \equiv 1$ (mod 12). The Steiner system constructed here will also contain $\frac{1}{12}v(v-1)$ subsystems of order $4v$. We observe that the direct product construction of an $S(2,3,v^2)$, an $S(2,4,v^2)$, respectively, provides Steiner systems with a completely different structure. We assume the reader is familiar with the Steiner system terminology and refer to [1, 7] for background and to [5, 6] for more literature on the subject.

The constructions

1. $S(2,3,v^2)$

Take any v pairwise disjoint STS(v)'s on the points $A_j, B_j, \ldots, Z_j$, $j = 1, 2 \ldots v$. Their blocks will be blocks of the STS(v^2) $\mathcal{S}$ we want to construct. Each of the remaining $\frac{1}{6}v^2v(v-1)$ blocks consists of three points in distinct STS(v)'s. To decide how to form triples of subsystems, we take any $STS(v)$, Σ, on the "points" $A, B, \ldots, Z$. For any block, ABC say, of Σ we use Construction 1 in [3], i.e. we construct an STS($3v$) having A,B, and C as subsystems. Thus the v^2 blocks coming from the triple ABC are obtained from the v base blocks [3]:

$$b_j = A_iB_jC_{2j-1} \qquad \text{for} \qquad j = 1, 2, \ldots, \frac{v+1}{2},$$

$$b_{j+\frac{(v+1)}{2}} = A_iB_{j+\frac{(v+1)}{2}}C_{2j} \qquad \text{for} \qquad j = 1, 2, \ldots, \frac{v-1}{2}.$$

Each base block yields v blocks, all containing the same point B_j, under the action of $\mathbb{Z}_v$ which acts on the subscript i of A_i by $i \to i+1 \pmod v$ and on the subscript h of C_h, $h = 2j-1$ and $h = 2j$, respectively, by $h \to h-1 \pmod v$. It is easy to check that the construction produces an STS(v^2) and, as soon as $v > 9$, one obtains many non-isomorphic STS(v^2)'s according to the choices of the subsystems of order v and to the system Σ.

2. $S(2,4,v^2)$

The construction is similar to that in 1. Take v pairwise disjoint $S(2,4,v)$'s on the points $A_j, B_j, \ldots, Z_j$, $j = 1,2\ldots v$, $v \equiv 1 \pmod{12}$. Their blocks will be blocks of the wanted $S(2,4,v^2)$ $\mathcal{S}$. The remaining $\frac{1}{12}v^2v(v-1)$ blocks of $\mathcal{S}$ all consist of four points in distinct $S(2,4,v)$'s. So, in order to form quadruples of such subsystems, take any $S(2,4,v)$, Σ, on the "points" $A, B, \ldots, Z$. For any block of Σ , say ABCD, construct the $S(2,4,4v)$ having A,B,C,D as subsystems using Construction 3 in [3]. This construction requires $v \equiv 1 \pmod{12}$, which takes care of our assumption. The v^2 blocks provided by the block ABCD of Σ are obtained by the action of $\mathbb{Z}_v$ on the following v base blocks [3]:

$$A_iB_jC_jD_{2j-1} \qquad \text{for} \qquad j = 1,2\ldots\frac{v+1}{2},$$

for each j, the subscript j of B_j is fixed and $\mathbb{Z}_v$ acts on the subscript i of A_i by $i \to i+1 (\mathrm{mod}\ v)$, on the subscript j of C_j by $j \to j-1 \pmod v$ and on the subscript $h = 2j-1$ of D_h by $h \to h-1 \pmod v$;

$$A_iB_{j+\frac{v+1}{2}}C_{j+\frac{v+1}{2}}D_{2j} \qquad \text{for} \qquad j = 1,2\ldots\frac{v-1}{2},$$

for each j, the subscript j of B_j is fixed and $\mathbb{Z}_v$ acts on the subscript i of A_i by $i \to i+1 \pmod v$, on the subscript $h = j + \frac{(v+1)}{2}$ of C_h by $h \to h-1 \pmod v$ and on the subscript $s = 2j$ of D_s by $s \to s-1 \pmod v$. It is easy to check that the construction indeed provides the required Steiner system.

We observe that any construction of an STS($3v$) with three disjoint subsystems of order v [4] can be used instead of Construction 1. Similarly, any construction of an $S(2,4,4v)$ containing four disjoint subsystems of order v can be used. Moreover, if such a construction can be used also when $v \equiv 4 \pmod{12}$, there is no unsettled case in the examined partition problem. (It should be possible to adapt some of the results in [2].)

References

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