A computer search for complex Golay sequences

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Abstract

We report on a computer search for complex Golay sequences of short length. All such sequences of length 11 and 13 will be classified.

1 Introduction

A *complex Golay sequence* (CGS) *of length* n is a pair

$$a_1a_2\dots a_n \quad b_1b_2\dots b_n$$

of $(\pm 1, \pm i)$-sequences which are complementary, that is, have zero autocorrelation:

$$0 = \sum_{j=1}^{k} a_j\bar{a}_{n-k+j} + \sum_{j=1}^{k} b_j\bar{b}_{n-k+j} \quad \text{for} \quad k = 1, 2, \dots, n-1.$$

It is convenient to denote 1 by $+$, -1 by $-$, i by i and $-i$ by j when displaying complex Golay sequences. Complex Golay sequences were introduced by Craigen in [1]. These sequences generalize real Golay sequences [4].

Given a CGS

$$a_1a_2\dots a_n \quad b_1b_2\dots b_n$$

an equivalent CGS is obtained by applying a finite number of the following operations.

Interchange:

$$b_1b_2\dots b_n \quad a_1a_2\dots a_n$$

Reflect both:

$$a_n \ldots a_2 a_1 \quad b_n \ldots b_2 b_1$$

Conjugate reflect either:

$$\overline{a}_n \ldots \overline{a}_2 \overline{a}_1 \quad b_1 b_2 \ldots b_n$$

$$a_1 a_2 \ldots a_n \quad \overline{b}_n \ldots \overline{b}_2 \overline{b}_1$$

Multiply either by $k = \pm 1, \pm i$:

$$(ka_1)(ka_2) \ldots (ka_n) \quad b_1 b_2 \ldots b_n$$

$$a_1 a_2 \ldots a_n \quad (kb_1)(kb_2) \ldots (kb_n)$$

Multiply by i^p where p is the position in the sequence:

$$(ia_1)(-a_2)(-ia_3)(a_4) \ldots (i^n a_n) \quad (ib_1) \ldots (i^n b_n)$$

An equivalence class can contain up to 1024 different sequences. For example, the equivalence class of $+ + - + \, i \, +$ $\quad + + - - \, j \, -$ has 1024 different sequences.

The following construction can be used to generate some CGS's from shorter length ones. The *double* of a CGS of length n

$$a_1 a_2 \ldots a_n \quad b_1 b_2 \ldots b_n$$

is the CGS of length $2n$

$$a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n \quad a_1 a_2 \ldots a_n (-b_1)(-b_2) \ldots (-b_n) \, .$$

For example, $+ +$ $\quad + -$ is equivalent to $+ +$ $\quad j \; i$ which doubles to $+ + \, j \; i$ $\quad + + \, i \; j$.

2 The search

We searched for CGS's of length up to 13. The sequences found are given in the last section.

Only sequences of form

$$+ a_2 \ldots a_{n-1} + \quad + b_2 \ldots b_{n-1} -$$

or

$$+ a_2 \ldots a_{n-1} \, i \quad + b_2 \ldots b_{n-1} \, j$$

were checked since any CGS is equivalent to one of these.

The search was further restricted by the following condition on the sequences. Let $a(+), \ldots, b(j)$ be the number of $+, \ldots, j$'s in the respective subsequences a_k and b_k. Then

$$(a(+)-a(-))^2+(a(i)-a(j))^2+(b(+)-b(-))^2+(b(i)-b(j))^2=2n.$$

To find the CGS's given below, we used a network of 18 Vaxstation 3100's, running programs written in the C language. These were used at lowest priority and did most of the computations. Each machine took about 4 hours to check 4^{15} sequences. A network of 5 SUN SPARCstation LX's and a DEC Alpha 3400s was also used intermittently. Each SUN took about 90 minutes, while the Alpha took under 60 minutes for 4^{15} sequences.

The computations were carried out in July, October and November of 1993. CGS's of length 11 required a day and a half, CGS's of length 12 required ten days, while CGS's of length 13 required about 70 days.

3 Applications

From complex Golay sequences of length n form circulant matrices G and H; see [1]. Then $GG^* + HH^* = 2nI_n$. Write $G = A + iB$ and $H = C + iD$. Then

$$AB^t - BA^t = CD^t - DC^t$$

and

$$AA^t + BB^t + CC^t + DD^t = 2nI_n.$$

Form ± 1 matrices $A+B$, $A-B$, $C+D$, $C-D$. A calculation shows similar formulas, where $2n$ is replaced by $4n$, hold for these four matrices. De Launey [2] shows that this gives cocyclic Hadamard matrices. Thus, if a complex Golay sequence of length n exists, then there is a cocyclic Hadamard matrix of order $4n$.

Corollary 1 *There are cocyclic Hadamard matrices of order* $4 \cdot 2^a 3^b 5^c 11^d 13^e$ *where* a, b, c, d *and* e *are non-negative integers, and* a *depends on* b, c, d *and* e.

In fact, the a in the Corollary is determined as follows. From the constructions available for generating CGS's, see [1], one can show that a CGS exists for all

$$a \geq b + d - 1 + [(c + e - b - d + 1)/2]$$

where $[x]$ is the largest non-negative integer greater or equal to x.

Corollary 2 *There are Hadamard matrices of order* $8n$ *where*

$$n = 2^{a_1}3^{b_1}5^{c_1}11^{d_1}13^{e_1} + 2^{a_2}3^{b_2}5^{c_2}11^{d_2}13^{e_2}$$

with a_i, b_i, c_i, d_i *and* e_i *non-negative integers, and* a_i *depending on* b_i, c_i, d_i *and* e_i *as above.*

The corollary shows that for values of n less than 1024 we obtain an order $8n$ Hadamard matrix with some exceptions. For 173 values of n including 87, 127, ..., 1023 the Hadamard matrix obtained is of order $16n$, and for 799, 927 and 967 the Hadamard matrix is of order $32n$. Further, the case 959 is open; however, no $n = 2^t \cdot 959$ for $0 \leq t \leq 5$ is of the above type, so if this corollary were to give a Hadamard matrix, its order would be at least $256 \cdot 959$.

Using an argument as above, from four supplementary $(\pm 1, \pm i)$-sequences of length n, one can construct eight supplementary circulant ± 1 matrices. So, by lemma 5 of [3], there are 16 supplementary Hermitian circulant matrices. One can then use an

$$\mathrm{OD}(2^{10}; \underbrace{2^6, \ldots, 2^6}_{16})$$

in 16 variables, see [3], to get a complex Williamson like matrix of order $2^{10}n$ constructible from 16 Hermitian matrices.

Corollary 3 *There is a Williamson like matrix of order* $2^{10}n$, *where* n *is as above.*

4 Complex Golay Sequences (up to equivalence) for lengths up to 13

In the following list of CGS's, the trailing D on some even length sequences indicates sequences which can be obtained by doubling CGS's of half the length.

Length 2: (64 sequences in total)

$$++ \qquad +-$$

Length 3: (128 sequences in total)

$$+\,i\,+ \qquad ++-$$

Length 4: (512 sequences in total)

$++-+$	$+++-$	D
$++ji$	$++ij$	D

Length 5: (512 sequences in total)

$+++ji$	$+i-+j$

Length 6: (2048 sequences in total)

$+++i-+$	$++j-+-$	
$++-+i+$	$++--j-$	D
$++-i-+$	$++j++-$	

Length 7:

none

Length 8: (6656 sequences in total)

$+++++--+$	$++--+-+-$	
$++++ijji$	$++--ijij$	
$+++-++-+$	$+++---+-$	D
$+++-++-+$	$+ii-+jj-$	
$+++-+-++$	$+++--+--$	D
$+++-iiji$	$+++-jjij$	D
$+++-ijii$	$+++-jijj$	D
$++-+---+$	$++-++++-$	D
$++-+jjji$	$++-+iiij$	D
$++---+-+$	$++++-++-$	
$++ii+-ji$	$++jj+-ij$	
$++iiji-+$	$++jjji+-$	
$++ij++ji$	$++ij--ij$	D
$++ij+-ii$	$++ij-+jj$	D
$++ijji++$	$++ijij--$	D
$++ijjj-+$	$++ijii+-$	D
$++jj-+ji$	$++ii-+ij$	

Length 9:

none

Length 10: (12288 sequences in total)

$+ + + + + - + - - +$	$+ + - - + + + - + -$	
$+ + + i + j i - j i$	$+ + j - - j + j i j$	
$+ + + i j + i j - i$	$+ + + i j - j i + j$	D
$+ + + i j + j - + i$	$+ + + i j - i + - j$	D
$+ + + i j - i - j i$	$+ + j - j + + j i j$	
$+ + + i j j - i j +$	$+ + + i j i + j i -$	D
$+ + + j + - i + j i$	$+ + i - j j - j i j$	
$+ + + j j j i + j i$	$+ + i - - + - j i j$	
$+ + - + - + - - + +$	$+ + - + + + + + - -$	
$+ + - + j + j j i i$	$+ + - + i + i i j j$	
$+ + - - i i - + - +$	$+ + + + i j - + + -$	
$+ + - i + - j - j i$	$+ + j + j j + i i j$	
$+ + - i - i - j i i$	$+ + - i + i + i j j$	
$+ + - i i i i - + +$	$+ + - i j i j + - -$	
$+ + - i j j j - j i$	$+ + j + - + + i i j$	
$+ + - j + j j + j i$	$+ + i + - j - i i j$	
$+ + - j j - j + j i$	$+ + i + j + - i i j$	
$+ + i i + j + - j i$	$+ + j j + i + - i j$	
$+ + i j i - i + + i$	$+ + i j i + j - - j$	D
$+ + j j i - - + j i$	$+ + i i i + - + i j$	

Length 11: (512 sequences in total)

$+ i - + - i j - i i +$	$+ + j j j + + i - + -$

Length 12: (36864 sequences in total)

$+ + + + - - - + i j - +$	$+ + i i + + - + + - + -$	
$+ + + + - - j i + - j i$	$+ + j j + + j i i j i j$	
$+ + + + - - j i - + j i$	$+ + i i + + j i i j i j$	
$+ + + + i + - j + - - +$	$+ + - - i + + i + - + -$	
$+ + + + i - + j + - - +$	$+ + - - i - - i + - + -$	
$+ + + - i + i - + - + +$	$+ + + - - j - i - + - -$	
$+ + + - i - i + + - + +$	$+ + + - + j + i - + - -$	
$+ + + - i i + - + + - +$	$+ + + - - - j i - - + -$	
$+ + + - i i - + + + - +$	$+ + + - + + j i - - + -$	
$+ + + i - + - + - i + +$	$+ + + i - + + - + j - -$	D
$+ + + i - + - - i + - +$	$+ + + i - + + + j - + -$	D
$+ + + i - + j i j - i i$	$+ + + i - + i j i + j j$	D
$+ + + i - + j j - i j i$	$+ + + i - + i i + j i j$	D
$+ + - + + + i j - - - +$	$+ + - + i i + - + + + -$	

+ + − + + − − i − + j +	+ + − − − + − i − − i −	
+ + − + + − j + j i − i	+ + − − − + j + j j + j	
+ + − + + − j − j i + i	+ + − − − + j − j j − j	
+ + − + − − i j − − − +	+ + − + i i − + + + + −	
+ + − + i + − − + + i +	+ + − + i + + + − − j −	D
+ + − + i + − i − − + +	+ + − + i + + j + + − −	D
+ + − + i + j − j j i i	+ + − + i + i + i i j j	D
+ + − + i + j j i i − i	+ + − + i + i i j j + j	D
+ + − − + i i + − + − +	+ + + + + i j − − + + −	
+ + − − − + + i + + i +	+ + − + + − + i + − j −	
+ + − − − − − + i j − +	+ + i i + + − + − + + −	
+ + − − − − j i + − j i	+ + j j + + j i j i i j	
+ + − − − − j i − + j i	+ + i i + + j i j i i j	
+ + − − − i i − − + − +	+ + + + − i j + − + + −	
+ + − − i − + + − + j +	+ + − − i − − − + − i −	D
+ + − − i − i i j i + i	+ + − − i − j j i j − j	D
+ + − i + i − j − j i i	+ + − i + i + i + i j j	D
+ + − i + i i − i − + +	+ + − i + i j + j + − −	D
+ + − i − + − + + i + +	+ + − i − + + − − j − −	D
+ + − i − + − − i − − +	+ + − i − + + + j + + −	D
+ + − i − + j i i − i i	+ + − i − + i j j + j j	D
+ + − i − + j j − j j i	+ + − i − + i i + i i j	D
+ + i + + − + + − j − +	+ + i + + − − − + i + −	D
+ + i + + − i i j + j i	+ + i + + − j j i − i j	D
+ + i − + − + + + j − +	+ + i − + − − − − i + −	D
+ + i − + − i i i + j i	+ + i − + − j j j − i j	D
+ + i i + + + − + − − +	+ + − − − − + − j i + −	
+ + i i + + + − − + − +	+ + + + − − + − j i + −	
+ + i i i + j + + − j i	+ + j j i + i − + − i j	
+ + i i i − i + + − j i	+ + j j i − j − + − i j	
+ + i j i + − j + − i i	+ + i j − j j − − + j j	
+ + i j i − − i + − i i	+ + i j + j i − − + j j	
+ + i j i i i j + + j i	+ + i j − − + − − − i j	
+ + i j i i j i + + j i	+ + i j + + + − − − i j	
+ + i j j j i j + + j i	+ + i j − − − + − − i j	
+ + i j j j j i + + j i	+ + i j + + − + − − i j	
+ + j j + i − i − + j i	+ + i i + i + j − + i j	
+ + j j − i − j − + j i	+ + i i − i + i − + i j	

Length 13: (512 sequences in total)

+ + + i − + + j + − + j i	+ i − − − i − + + j − + j

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