A few more BIBDs with $k = 8$ or 9

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Abstract. It has been shown that for $k = 8$ or 9, there exists a BIBD$(v, k, 1)$ for all positive integers $v \equiv 1$ or $k \pmod{k(k-1)}$, with some possible exceptions. We show that such designs exist for 49 of these exceptional values.

1. Introduction

A balanced incomplete block design (BIBD) with parameters $(v, k, 1)$ is a pair $(X, \mathbb{A})$ where X is a v-set and $\mathbb{A}$ is a family of k-subsets (where $2 < k < v$) called *blocks*, such that every pair of distinct points of X occurs in exactly one block of $\mathbb{A}$.

It is well known that

(i) $(v-1) \equiv 0 \pmod{k-1}$, and
(ii) $v(v-1) \equiv 0 \pmod{k(k-1)}$

are necessary conditions for the existence of a BIBD$(v, k, 1)$.

For $k = 8$ or 9, these conditions reduce to the condition that v be congruent to 1 or $k \pmod{k(k-1)}$. In previous papers [3,4] it has been shown that for any positive integer $v \equiv 1$ or $k \pmod{k(k-1)}$ there exists a BIBD$(v, k, 1)$ with some possible exceptions.

It is our purpose here to reduce this number of possible exceptions to 64 for $k = 8$ and 122 for $k = 9$.

To construct the designs that eliminate these exceptions, we require other combinatorial configurations. For the definitions of pairwise balanced design (PBD), transversal design (TD), group divisible design (GDD), PBD-closed set, and the various composition constructions for PBDs, see [6]. We also adopt the notation of this reference.

We adopt the notation $B(K) = \{v :$ a PBD$(v, K, 1)$ exists$\}$ and $R_k = \{r :$ a BIBD$((k-1)r+1, k, 1)$ exists$\}$. We denote by PBD$(v, K \cup \{q^*\}, 1)$ a PBD which has exactly one block of size q and all other block sizes in K. We use the notation $v \in B(K \cup \{q^*\})$ to indicate the existence of a PBD$(v, K \cup \{q^*\})$.

In what follows, we shall first investigate the case $k = 8$, and then $k = 9$.

2. The case $k = 8$

We need the following lemma, which is essentially Lemma 4.17 in [3].

Lemma 2.1. *Suppose*

(1) a $\mathrm{TD}(10, t)$ *exists and* $7t + q \in B(R_8 \cup \{q^*\})$;
(2) $7u + q \in B(R_8 \cup \{q^*\}), \quad 0 \le u \le t$;
(3) $8v + q \in R_8, \quad 0 \le v \le t$.

Then $r = 56t + 7u + 8v \in R_8$.

We then have

Corollary 2.2. $\{544, 552, 608\} \subset R_8$.

Proof. The conclusion comes from Lemma 2.1 and the expressions

$$544 = 56 \times 9 + 7 \times 1 + 8 \times 4 + 1$$
$$552 = 56 \times 9 + 7 \times 1 + 8 \times 5 + 1$$
$$609 = 56 \times 9 + 7 \times 9 + 8 \times 5 + 1.$$

The existence of the $\mathrm{TD}(10, t)$ comes from [1], the others from [4].

From Theorem 1.2 in [3] we have

Lemma 2.3. $\{152, 560, 600, 616, 624, 1008, 1392, 1400, 1448, 1456, 1504\} \subset R_8$.

We obtain the following theorem. (This updates the result in [4].)

Theorem 2.4. *A BIBD$(v, 8, 1)$ exists whenever $v \equiv 1$ or 8 (mod 56), with 64 possible exceptions for $v = 7r + 1$, where r is shown in Table 1.*

Table 1

16	24	25	32	40	48	56	88	89	96
104	105	112	160	161	168	176	177	184	185
192	200	201	208	209	216	217	224	225	248
256	272	280	304	312	320	360	368	376	384
416	424	472	480	488	496	536	768	808	816
824	880	952	1528	1529	1536	1560	1568	1576	1584
1592	1784	1840	1848						

3. The case $k = 9$

In this section we construct BIBDs with $k = 9$.

Since a BIBD(153, 9, 1) exists from Appendix I in [5], we have

Lemma 3.1. $E_1 = \{19, 153\} \subset R_9$.

We then have

Lemma 3.2. $E_2 = \{163, 361\} \subset R_9$.

Proof. Adding a set of 9 new points to each group of a TD(9, 144) and using the existence of a BIBD(153, 9, 1), we obtain a BIBD(1305, 9, 1) and $163 \in R_9$. $361 \in R_9$ comes from TD(19,19) and $19 \in R_9$.

For our recursive constructions, we need some GDDs.

Lemma 3.3. *There exist* $\{9, 10\}$*-GDDs of the following group-types:*

$$(a)\ 8^9, \qquad (b)\ 8^{10}, \qquad (c)\ 8^9 9^1, \qquad (d)\ 9^{10}, \qquad (e)\ 9^{10}10^1.$$

Proof. (a), (b) and (d) are fairly obvious. For (c), we adjoin a new point to a TD(9,9) to obtain a PB(82, $\{9, 10\}$, 1), and then delete an old point from the block of size 10. For (e), we delete one block of a TD(10,11) and the result follows.

Now we give some constructions; all their proofs are similar.

Lemma 3.4. *Suppose*

(1) a $TD(10, t)$ *exists, and* $8t + q \in B(R_9 \cup \{q^*\})$;

(2) $8u + 9v + q \in R_9, \quad 0 \le u + v \le t$.

Then $r = 72t + 8u + 9v + q \in R_9$.

Proof. In all groups but one of a TD(10, t), we give the points weight 8. In the last group, we give u points weight 8, v points weight 9, and give the remaining points weight 0. We can apply Wilson's construction from [6] with the necessary input designs from Lemma 3.3 to obtain a $\{9, 10\}$-GDD of group type $(8t)^9(8u + 9v)^1$. We then adjoin a set of q new points to the groups of this GDD, using the facts that $8t + q \in B(R_9 \cup \{q^*\})$ and $8u + 9v + q \in R_9$ to obtain the desired result.

Corollary 3.5. $E_3 = \{1315, 1324, 1504\} \subset R_9$.

Proof. The conclusion comes from Lemma 3.4 and the following expressions:

$$1315 = 72 \times 17 + 8 \times 8 + 9 \times 3 + 0$$
$$1324 = 72 \times 17 + 8 \times 8 + 9 \times 4 + 0$$
$$1504 = 72 \times 19 + 8 \times 0 + 9 \times 15 + 1.$$

The existence of the TD(10, t)s comes from [1], the others from [4] and Lemma 3.1.

Lemma 3.6. *Suppose*

(1) *a TD$(11,t)$ exists, and* $9t+q \in B(R_9 \cup \{q^*\})$;

(2) $10u+q \in R_9$, $0 \le u \le t$.

Then $r = 90t + 10u + q \in R_9$.

Corollary 3.7. $E_4 = \{1071, 1720, 2800\} \subset R_9$.

Proof. The conclusion comes from Lemma 3.6 and the following expressions:

$$1071 = 90 \times 11 + 10 \times 8 + 1$$
$$1720 = 90 \times 19 + 10 \times 1 + 0$$
$$2800 = 90 \times 29 + 10 \times 19 + 0.$$

The existence of the TD$(11,t)$s comes from [1], the others from [4].

Lemma 3.8. *Suppose*

(1) *a TD$(10,t)$ exists, and* $t+q \in B(R_9 \cup \{q^*\})$;

(2) $u+q \in R_9$, $0 \le u \le t$.

Then $r = 9t + u + q \in R_9$.

Corollary 3.9. $E_5 \subset R_9$, *where*

$$E_5 = \{172, 180, 181, 919, 1558, 1567, 1584, 1648, 1719, 1728, 1729, 1746, 1755, 1764, 1773, 1809, 1846, 1854, 1881, 1882, 2134, 2205, 2278, 3348, 4087\}.$$

Proof. We apply Lemma 3.8 with the parameters shown in Table 2. The existence of the TD$(10,t)$s (except TD(10,189)) comes from [1]; TD(10,189) comes from [2].

Table 2

r	t	u	q	r	t	u	q
172	19	1	0	1764	179	152	1
180	19	9	0	1773	179	161	1
181	19	10	0	1809	181	180	0
919	100	19	0	1846	189	144	1
1558	163	91	0	1854	189	152	1
1567	163	100	0	1881	189	179	1
1584	163	117	0	1882	189	180	1
1648	181	19	0	2134	217	181	0
1719	181	90	0	2205	225	180	0
1728	179	116	1	2278	233	180	1
1729	181	100	0	3348	352	180	0
1746	181	117	0	4087	414	361	0
1755	179	143	1				

It has been shown that $\bigcup_{1 \le i \le 5} E_i \subset R_9$. We obtain the following theorem. (This updates the result in [4].)

Theorem 3.10. *A BIBD$(v,9,1)$ exists whenever $v \equiv 1$ or 9 (mod 72), with 122 possible exceptions for $v = 8r+1$, where r is shown in Table 3.*

Table 3

18	27	28	36	37	45	46	54	63	72
99	108	109	118	126	127	135	189	198	199
208	216	226	235	243	244	253	279	280	288
298	306	307	333	334	342	343	360	370	378
387	388	405	415	423	424	433	450	468	469
477	478	504	531	532	540	541	549	550	558
559	567	603	604	612	613	648	675	684	685
693	702	756	766	774	783	828	829	837	838
846	847	864	873	918	928	954	963	1017	1027
1035	1036	1152	1188	1189	1197	1242	1252	1269	1278
1323	1333	1350	1359	1414	1422	1495	1566	1593	1674
1819	1827	1828	1836	1908	1935	1971	1998	2062	2071
2223	2386								

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