ON MAXIMAL SETS OF ONE-FACTORS

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ABSTRACT: A set S of edge-disjoint one-factors in a Graph G is said to be maximal if there is no one-factor of G which is edge-disjoint from S, and if the union of S is not all of G. Maximal sets of one-factors of K_{2n} have been investigated and until very recently only results for particular cases have been obtained. In this paper we present a new technique for solving the problem.

1. INTRODUCTION

We consider graphs which are undirected, finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [1]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices and $\varepsilon(G)$ edges. K_n denotes the complete graph on n vertices and $K_{n,m}$ denotes the complete bipartite graph with bipartitioning sets of size n and m.

A **1-factor** of a graph G is a 1-regular spanning subgraph. A **1-factorization** of G is a set of (pairwise) edge-disjoint one factors which between them contain each edge of G. It is very well known (see [3]) that K_{2n} and $K_{n,n}$ have 1-factorizations for all n.

A set F of edge disjoint 1-factors in a graph G is said to be **maximal** if there is no 1-factor which is edge-disjoint from F and if F

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is not all of G. Thus if we write $\bar{F}$ for the complement in G of the union of members of F, then F is maximal if and only if $\bar{F}$ is a non-empty graph with no 1-factor. We call $\bar{F}$ the **leave** of F. Observe that if G is regular, then $\bar{F}$ is regular. If $\bar{F}$ is d-regular, then F is called a maximal set of **deficiency** d or simply a **d-set**. The existence of d-sets in K_{2n} for $n > 2$ was shown by Cousins and Wallis [4].

Caccetta and Wallis [2] established that 3-sets exist in K_{2n} for every $2n \geq 16$. This was accomplished by first establishing properties which reduced the problem to one of finding 3-sets in K_{2n} for $16 \leq 2n \leq 28$, and then exhibiting the required 3-sets. In this paper we generalize these methods. In particular, we prove that if K_{2n} has a d-set, then K_{4n-2t} has a d-set for each $0 \leq t \leq n - \frac{1}{2}(d + 1)$. We apply this result to show that 5-sets exist in K_{2n} for every $2n \geq 22$.

Recently, Rees and Wallis [6] solved the problem of determining the spectrum of maximal sets of 1-factors in K_{2n}. Our approach is, however, quite different and has the potential to yield a simpler and more intuitive proof. Our main result is of interest in its own right.

2. PRELIMINARIES

In this section we discuss three results which we make use of in the proof of our main theorem. A **matching** M in a graph G is a subset of E(G) in which no two edges have a common vertex. We begin by stating a lemma proved in Rees and Wallis [6].

Lemma 2.1. Let $K_{m,n}$ be the complete bipartite graph with bipartition (X,Y), where $|X| = m$, $|Y| = n$ and $m \leq n$. Let $Y_1, Y_2, \ldots, Y_n$ be any collection of m-subsets of Y such that each vertex $y \in Y$ is contained in exactly m of the Y_j's. Then there is an edge-decomposition of $K_{m,n}$ into matchings $M_1, M_2, \ldots, M_n$ where for each $j = 1,2,\ldots,n$ M_j is a matching with m edges from X to Y_j. □

The **edge-chromatic number** $\chi'(G)$ of a graph G is the minimum number of colours needed to colour the edges of G. Our next lemma is a special case of a theorem of Folkman and Fulkerson [5]. The proof we give was given to us in a personal communication by Rees.

Lemma 2.2. If G is a graph with c.k edges and $c \geq \chi'(G)$, then the edge set of G admits a decomposition into c matchings, each with k edges.

Proof: Let $\mathcal{C}$ be the set of all proper c-colourings of G. Note that $\mathcal{C} \neq \phi$ since $c \geq \chi'(G)$. For $K \in \mathcal{C}$, define

$$n(K) = \sum_{i=1}^{c} |e_i - k| ,$$

where e_i is the number of edges in the i^{th} matching (i.e. i^{th} colour class) of K, $i = 1,2,\ldots,c$.

Let

$$n_0 = \min\{n(K): K \in \mathcal{C}\} ,$$

and let K_0 be a colouring for which $n(K_0) = n_0$. We will prove that $n_0 = 0$, i.e. K_0 is a decomposition of G into c matchings, each with k edges. Suppose that this is not the case and $n(K_0) > 0$. Then there is a matching M_i for which $e_i = |M_i|$ is not k. Now since $\varepsilon(G) = ck$, there must be matchings M_1 and M_2 say, with $e_1 = |M_1| < k$ and $e_2 = |M_2| > k$.

Let H be the subgraph of G whose edge set is $M_1 \cup M_2$. Then H is the disjoint union of cycles and paths. Since $e_2 > e_1$, H must contain as a component a path P of odd length which begins and ends with an edge of M_2. Now switch the colours in P, i.e. those edges of P that were coloured 1 get coloured 2 and vice-versa. Let us call the matchings corresponding to these colour changes M_1' and M_2'. This creates a new colouring K_0' of G with corresponding matchings $M_1', M_2', M_3, \ldots, M_c$. Furthermore,

$$e_1' = |M_1'| = e_1 + 1 ,$$

and

$$e_2' = |M_2'| = e_2 - 1.$$

Now recalling that $e_1 < k$ and $e_2 > k$, we have

$$|e_1' - k| < |e_1 - k| ,$$

and

$$|e_2' - k| < |e_2 - k| .$$

Hence

$$n(K_0') < n(K_0) ,$$

and this contradicts the minimality of $n(K_0)$. It thus follows that $n_0 = 0$. This proves the lemma. □

We conclude this section by stating a result of Wallis [7].

Lemma 2.3. A d-regular graph G with no 1-factor and no odd-component satisfies:

$$\nu(G) \geq \begin{cases} 3d + 7, & \text{for odd } d \geq 3 \\ 3d + 4, & \text{for even } d \geq 6 \\ 22, & \text{for } d = 4 . \end{cases}$$

No such G exists for d = 1 or 2. □

3. MAIN RESULT

Our main result is essentially a generalization of Theorems 4 and 5 of Caccetta and Wallis [2].

Theorem 3.1. Suppose for odd d there exists a d-set in K_{2n}. Then for each $0 \leq t \leq n - \frac{1}{2}(d + 1)$ there is a d-set in K_{4n-2t}.

Proof: We can write $K_{4n-2t} = K_{2n-2t} \vee K_{2n}$. Let X and Y denote the graphs K_{2n-2t} and K_{2n}, respectively. Now Y has a maximal set of (2n - d - 1) 1-factors. Take 2t of these 1-factors and let H be the graph formed by the union of these 1-factors.

Applying Lemma 2.2 (with $c = 2n$ and $k = t$) we decompose the edge-set of H into 2n matchings $M_1, M_2, \ldots, M_{2n}$, each with t edges. Let Y_i denote the vertices of Y not saturated by the matching M_i. Note that since H has regularity 2t, each vertex in Y will be contained in exactly 2n-2t of the Y_i's. Furthermore, each Y_i contains exactly 2n-2t vertices of Y.

Now we apply Lemma 2.1 to the subgraph $K_{2n-2t,2n}$. This yields 2n disjoint matchings $N_1, N_2, \ldots, N_{2n}$, where N_i joins the vertices of Y_i to the vertices of X. Let

$$L_i = M_i \cup N_i \qquad i = 1,2,\ldots,2n.$$

There remain in Y a set S of $(2n - 1 - d) - 2t$ 1-factors from the original maximal set on Y. Construct $(2n - 1 - d) - 2t$ 1-factors on X (such a set exists since K_{2p} has a 1-factorization) and pair these off with the 1-factors of S to form a set of $(2n - 1 - d - 2t)$ 1-factors $\bar{L}_1, \bar{L}_2, \ldots, \bar{L}_{2n-1-d-2t}$. Then the set

$$F = \{L_i:\ i = 1,2,\ldots,2n\} \cup \{\bar{L}_j:\ j = 1,2,\ldots,2n-1-d-2t\}$$

forms a maximal set of 1-factors of deficiency d in K_{4n-2t}. Note that the leave $\bar{F}$ of F consists of 2-components one of which is the leave of the maximal set of 1-factors in K_{2n}. This completes the proof of the theorem. □

As a corollary we have:

Corollary: If K_{2n} has a d-set, d odd, then for each even integer $m \geq 2n + d + 1$, K_m has a d-set.

Proof: Suppose K_{2n} has a d-set, d odd. Then by Theorem 3.1 there exists a d-set in K_{2n+d+1}, K_{2n+d+3}, $\ldots$, K_{4n}. Further a d-set in K_{2n+d+1} implies a d-set in $K_{2n+2d+2}$, $K_{2n+2d+4}$, $\ldots$, $K_{4n+2d+2}$. Now since a d-set in K_{2n} implies (Dirac's Theorem) that $d \leq n$ we have $2n + 2d + 2 \leq 4n + 2$. Hence repeated applications of Theorem 3.1 will in fact cover all even integers $m \geq 2n + d + 1$. This completes the proof of the Corollary. □

4. APPLICATION OF THEOREM 3.1

We now discuss the application of Theorem 3.1. First we consider the existence of 3-sets in K_{2n}. Since, by Lemma 2.3, the smallest 3-regular graph without a 1-factor contains at least 16 vertices, K_{2n} has no 3-set for $2n \leq 14$. A 3-set in K_{16} was shown in [2]. The above result implies that if we can find a 3-set in K_{18}, then we have a 3-set in K_{2n} for every $2n \geq 16$. This is the case as shown in [2]. We remark that the proof that K_{2n} has a 3-set for every $2n \geq 16$ in [2] involved the construction of 3-sets in K_{2n} for $16 \leq 2n \leq 28$. Application of Theorem 3.1 eliminates the need to look at the cases $20 \leq 2n \leq 28$.

We now illustrate the work involved in establishing the existence of d-sets, by consider the case $d = 5$.

Lemma 2.3 implies that 5-sets do not exist in K_{2n} for $2n \leq 20$. So suppose $2n \geq 22$. We will exhibit 5-sets in K_{22}, K_{24} and K_{26}. Then the corollary to Theorem 3.1 implies the existence of 5-sets in K_{2n} for every $2n \geq 22$.

Consider K_{22} with vertices labelled 1,2,...,9, A,B,...,M. Take the 16 1-factors:

T_1	=	18	25	3D	4L	JC	6H	7A	9I	BF	EK	MG
T_2	=	15	2G	3E	4I	8H	6A	7J	9F	BK	CM	DL
T_3	=	19	2E	3L	4H	5D	6J	7G	8K	AF	BM	CI
T_4	=	1A	2H	3F	4M	5C	6I	7K	8L	9D	BG	EJ
T_5	=	1B	2F	3K	4C	5I	6D	7L	8J	9M	AH	GE
T_6	=	1C	29	3H	4J	5K	6F	7E	8G	DI	AM	BL
T_7	=	1D	2I	3M	4F	5A	6G	7H	8C	9K	BJ	EL
T_8	=	1E	2C	3J	48	5B	69	7D	AI	LH	MF	KG
T_9	=	1F	2L	3G	4E	59	6M	7C	8I	AJ	DK	BH
T_{10}	=	1G	2M	38	47	5E	6B	LI	AK	CH	DF	9J
T_{11}	=	1H	2J	3A	4K	5G	6C	7M	8F	9L	BI	DE
T_{12}	=	1I	2A	36	4B	58	HE	7F	LC	9G	DJ	KM
T_{13}	=	1J	28	3C	49	5L	DG	7I	AB	FE	HM	K6
T_{14}	=	1K	2B	79	4A	5F	6L	3I	8E	CG	DH	JM
T_{15}	=	1L	2K	3B	4D	5J	6E	78	9H	AG	FC	MI
T_{16}	=	1M	2D	39	4G	5H	68	7B	CK	AL	FJ	EI

The leave of this set of 1-factors is given in Figure 4.1. Thus we have a 5-set in K_{22}.

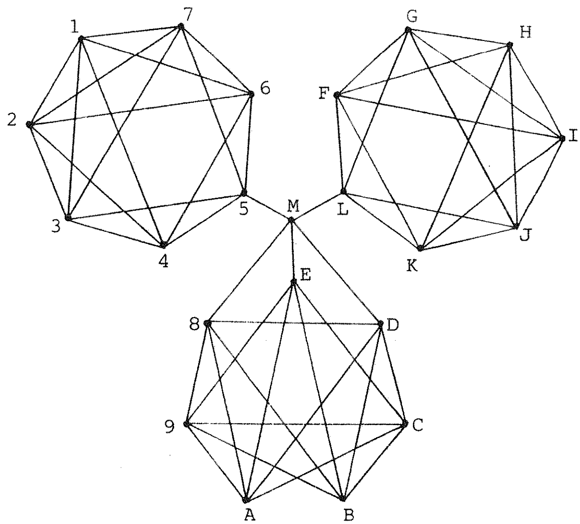

Figure 4.1

Consider K_{24} with vertices labelled 1,2,...,9, A,B,...,O. Take the 18 1-factors:

R_1	=	14	2J	36	DI	5G	8F	7E	NO	9H	AL	BM	CK
R_2	=	16	2D	30	4B	58	9L	7C	JN	AE	FG	HK	MI
R_3	=	17	25	3L	48	6M	9K	AN	BF	CD	EH	GI	JO
R_4	=	1A	2N	3M	47	59	6J	8C	BL	DH	EO	FI	GK
R_5	=	1B	20	3N	4M	5L	69	7G	8A	CH	DJ	EI	FK
R_6	=	10	2K	3I	4L	5B	6C	7M	8D	9A	GJ	EN	FH
R_7	=	1D	2G	3E	4F	5I	6N	7H	8M	9C	AK	BJ	LO
R_8	=	1C	2F	3H	4G	50	6K	7N	8I	9B	AM	EJ	DL
R_9	=	1E	2H	3F	40	5J	6G	7K	8L	9D	AI	BN	CM
R_{10}	=	1L	2A	37	4K	5C	6E	8J	9F	BH	DM	GN	IO
R_{11}	=	1G	2B	3A	4J	5H	6D	7I	8K	9E	CL	MO	FN
R_{12}	=	1H	2L	3J	4A	5E	6B	7D	8G	9N	CI	FM	KO
R_{13}	=	1I	26	39	4C	5D	70	8N	AH	BK	EL	FJ	GM
R_{14}	=	1J	28	3K	4I	5F	6A	7B	9G	CO	DN	EM	HL
R_{15}	=	1K	2M	CN	4E	5A	6H	7F	80	9J	GL	3D	BI
R_{16}	=	1F	2C	3B	DO	6L	7J	9I	5M	AG	EK	4N	8H
R_{17}	=	1M	2E	3G	DK	5N	FL	7A	8B	90	4H	6I	CJ
R_{18}	=	1N	2I	3C	4D	5K	6F	7L	8E	9M	HG	AJ	BO

The leave of this set of 1-factors is given in Figure 4.2. We thus have a 5-set in K_{24}.

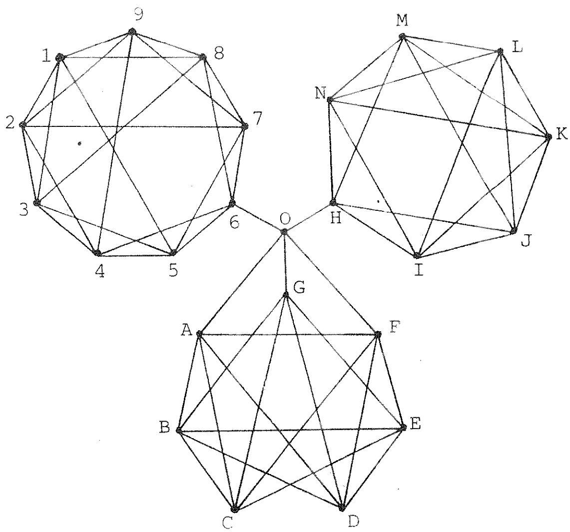

Figure 4.2

Finally, consider K_{26} with vertices labelled 1,2,...,9, A,B,...,Q. A suitable 5-set is:

T_1 =	14	2G	3Q	5L	6J	7M	8C	9H	AO	BF	DP	EK	NI
T_2 =	15	27	39	4C	6B	8A	DK	EQ	FN	GM	HL	IO	JP
T_3 =	16	2A	3C	49	50	7D	8B	EP	FI	GK	HN	JM	LQ
T_4 =	17	28	36	4B	5A	9C	DE	FH	GJ	IL	KN	MP	OQ
T_5 =	1A	2D	3B	47	5I	6N	8F	9P	CO	EJ	GL	HK	MQ
T_6 =	1B	2M	3H	4L	5K	6P	7A	8E	9J	CQ	DI	FO	GN
T_7 =	1C	2L	3J	4G	5P	6M	7H	8I	9B	AK	DN	EO	FQ
T_8 =	1D	2F	3N	4K	5E	6H	7L	80	9A	BI	CM	GP	JQ
T_9 =	1E	2B	38	4A	5N	6L	7C	DM	9K	FP	GH	IQ	JO
T_{10}=	1F	2Q	3D	4P	5J	60	7G	8K	9I	AB	CL	EN	HM
T_{11}=	1G	20	3P	4D	5C	6F	7J	8M	9N	BL	AH	EI	KQ
T_{12}=	1H	2E	3A	4N	5Q	6G	70	8P	9M	BK	CI	DL	FJ
T_{13}=	1I	2N	3M	40	58	6A	7E	9G	BH	CJ	DQ	FL	KP
T_{14}=	1J	2K	3G	4E	5B	6I	7P	8H	9Q	AL	CN	DO	FM
T_{15}=	1K	2J	30	4Q	5F	6D	7N	8L	9E	AP	BM	CH	GI
T_{16}=	1L	2H	3E	4I	5M	6K	7F	8Q	9D	AN	CP	BJ	GO
T_{17}=	1M	2P	3L	4F	5D	6Q	7I	8G	90	AJ	BN	CK	EH
T_{18}=	1N	2I	3K	4J	5H	6C	7Q	8D	9F	AG	BP	EM	LO
T_{19}=	10	2C	3I	4H	5G	69	7B	8N	AM	EL	DJ	FK	PQ
T_{20}=	1P	25	3F	4M	DH	6E	7K	8J	9L	AI	BO	CG	NQ

The leave of this set is given in Figure 4.3.

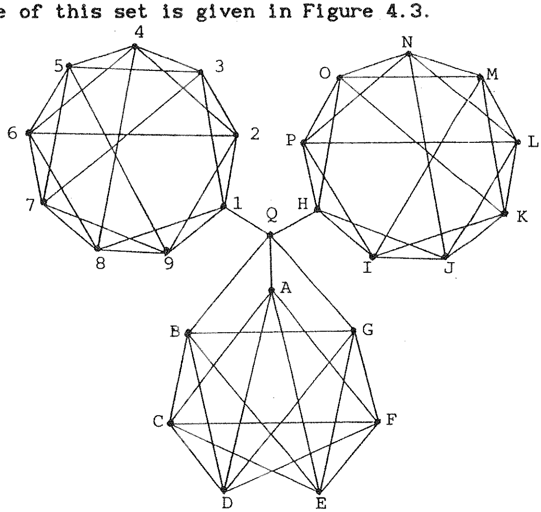

Figure 4.3

We have proved:

Theorem 4.1. There exists a 5-set in K_{2n} for every $2n \geq 22$. □

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