A NOTE ON THE CYCLE INDEX POLYNOMIAL OF THE SYMMETRIC GROUP

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Abstract: An identity concerning the cycle index polynomial of the symmetric group is proved and a consequence of it presented.

INTRODUCTION

Let X be a set with n indistinguishable elements and let k be a positive integer. Let $T(n,k)$ denote the number of ways of choosing k subsets of X whose union is X. Let $\mathcal{P}_n$ denote the set of all partitions of n. If $\lambda \in \mathcal{P}_n$ we write $\lambda = (\lambda_1, \ldots, \lambda_r)$, where $n = \lambda_1 + \cdots + \lambda_r$ and $r = r(\lambda)$ is the number of parts in λ. In [2], the following two expressions for $T(n,k)$ were obtained.

$$T(n,k) = \frac{1}{n!\,k!} \sum_{\substack{\lambda \in \mathcal{P}_n \\ \mu \in \mathcal{P}_k}} \begin{bmatrix} n \\ \lambda \end{bmatrix} \begin{bmatrix} k \\ \mu \end{bmatrix} \prod_i \left(\left(\prod_j 2^{(\lambda_i, \mu_j)} \right) - 1 \right). \tag{1}$$

$$T(n,k) = U(n,k) - U(n-1,k), \tag{2}$$

where

$$U(n,k) = \frac{1}{n!\,k!} \sum_{\substack{\lambda \in \mathcal{P}_n \\ \mu \in \mathcal{P}_k}} \begin{bmatrix} n \\ \lambda \end{bmatrix} \begin{bmatrix} k \\ \mu \end{bmatrix} \prod_{i,j} 2^{(\lambda_i, \mu_j)}. \tag{3}$$

Here (a,b) denotes the greatest common divisor of a and b and $\begin{bmatrix} n \\ \lambda \end{bmatrix}$ is the number of permutations in the symmetric group S_n with cycle type λ.

It is claimed in [2] that the equality of (1) and (2) is equivalent to the following result.

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Theorem 1. *For any m-tuple of positive integers $(a_1, \ldots, a_m)$ and any positive integer n,*

$$\sum_{\lambda \in \mathcal{P}_n} \begin{bmatrix} n \\ \lambda \end{bmatrix} \prod_i \left(\left(\prod_{j=1}^{m} 2^{(\lambda_i, a_j)} \right) - 1 \right) = \sum_{\lambda \in \mathcal{P}_n} \begin{bmatrix} n \\ \lambda \end{bmatrix} (1 - s_1(\lambda)/2^m) \prod_{i,j} 2^{(\lambda_i, a_j)},$$

where $s_1(\lambda)$ denotes the number of parts of λ of size 1.

Now it is easy to show that Theorem 1 *implies* the equivalence of (1) and (2), but it is not easy to deduce the Theorem from these two results. However, in this note we deduce Theorem 1 from an old result of Bell on the cycle index polynomial of the symmetric group.

THE CYCLE INDEX POLYNOMIAL

Let n be a positive integer. The *cycle index polynomial* [Polya, 5] of S_n is the polynomial

$$f_n(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\lambda \in \mathcal{P}_n} \begin{bmatrix} n \\ \lambda \end{bmatrix} \prod_{i=1}^{r(\lambda)} x_{\lambda_i}.$$

Lemma.

$$f_n(x_1 - 1, \ldots, x_n - 1) = f_n(x_1, \ldots, x_n) - \frac{\partial}{\partial x_1} f_n(x_1, \ldots, x_n)$$

.

Proof. Write $f_n(\overline{x}) = f_n(x_1, \ldots, x_n)$. From Bell [1, page 265] or Riordan [6, page 80] we have that

$$f_n(\overline{x} + \overline{y}) = \sum_{j=0}^{n} f_{n-j}(\overline{x}) f_j(\overline{y}).$$

Setting $\overline{y} = (-1, -1, \ldots, -1)$, we obtain

$$f_n(x_1 - 1, \ldots, x_n - 1) = f_n(x_1, \ldots, x_n) - f_{n-1}(x_1, \ldots, x_{n-1}).$$

By Riordan [6, page 70], $f_{n-1} = \frac{\partial f_n}{\partial x_1}$. Hence the result follows.

Proof of Theorem 1. The result of the above Lemma may be written as

$$\sum_{\lambda \in \mathcal{P}_n} \begin{bmatrix} n \\ \lambda \end{bmatrix} \prod_i (x_{\lambda_i} - 1) = \sum_{\lambda \in \mathcal{P}_n} \begin{bmatrix} n \\ \lambda \end{bmatrix} (1 - s_1(\lambda)/x_1) \prod_i x_{\lambda_i}.$$

We obtain Theorem 1 by substituting $x_1 = 2^m$ and $x_i = \prod_j 2^{(i,a_j)}$ for $i > 1$ in this equation.

It follows immediately that Theorem 1 may be generalised by replacing 2 by any real number r.

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