Block-transitive designs and maximal subgroups of finite symmetric groups.

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1. Introduction

A $t-(v, k, \lambda)$ design is a pair $\mathcal{D} = (X, \mathcal{B})$, where X is a set of v points, and $\mathcal{B}$ is a set of k-element subsets of X called blocks, such that any t points are contained in exactly λ blocks, where $\lambda > 0$. Such a design is called trivial if $\mathcal{B}$ consists of all the k-element subsets of X . An automorphism of a design $\mathcal{D}$ is a permutation of the point set X which fixes $\mathcal{B}$ setwise (in its induced action on k-element subsets of X). In this paper we discuss some construction methods for block-transitive t-designs, that is for t-designs $\mathcal{D}$ for which the group of automorphisms of $\mathcal{D}$ is transitive on the block set $\mathcal{B}$. Let $\mathcal{D} = (X, \mathcal{B})$ be a $t-(v,k,\lambda)$ design with automorphism group G . By a result of R.E. Block [1] the number of G-orbits in $\mathcal{B}$ is greater than or equal to the number of G-orbits in X . In particular if G is block-transitive then G is also point-transitive, that is G is a transitive subgroup of the symmetric group Sym (X) on X . Suppose now that $\mathcal{D}$ is block-transitive. It was shown in [2, Proposition 1.1] that, for any over-group H of G in Sym (X), the possibly larger family $\mathcal{B}^* = \{B^h \mid B \in \mathcal{B}, h \in H\}$ of k-element subsets of X is also the block set of a $t-(v,k,\lambda^*)$ design $\mathcal{D}^* = (X, \mathcal{B}^*)$ for some $\lambda^* \geq \lambda$. The design $\mathcal{D}^*$ is also block-transitive with automorphism group containing H . If H is k-homogeneous on X, that is if H is transitive on the k-element subsets of X, then $\mathcal{D}^*$ will be a trivial design, while if H is not k-homogeneous on X then $\mathcal{D}^*$ will be nontrivial. Consider the problem:

Problem. Given positive integers t, v, k, decide whether there exists a nontrivial block-transitive $t-(v,k,\lambda)$ design for some $\lambda > 0$.

According to our discussion, one way of deciding this is to check, for each maximal non-k-homogeneous subgroup H of $\mathrm{Sym}(X)$ and for each H-orbit $\mathcal{B}$ on k-element subsets, whether $(X,\mathcal{B})$ is a t-design. This decision can be made by examining a single k-subset B of $\mathcal{B}$ as follows: Let $Q_1, \ldots, Q_m$ be the H-orbits on t-element subsets of X, and for each $i = 1, \ldots, m$ let q_i be the number of t-element subsets of B which belong to Q_i. Then, by [2, Proposition 1.3], $(X,\mathcal{B})$ is a t-design if and only if

$$\frac{q_1}{|Q_1|} = \frac{q_2}{|Q_2|} = \cdots = \frac{q_m}{|Q_m|} . \tag{1}$$

According to the O'Nan-Scott Theorem (see [5]) the maximal transitive subgroups G of S_v are of one of the following types:

1. imprimitive: $G = S_c \text{ wr } S_d$, where $v = cd$, $c > 1$, $d > 1$;
2. affine: $G = AGL(d,p)$, where $v = p^d$, p is a prime and $d \geq 1$;
3. product: $G = S_c \text{ wr } S_d$, where $v = c^d$, $c \geq 5$, $d > 1$;
4. simple diagonal: $G = T^d(\mathrm{Out}\, T \times S_d)$, where $v = |T|^{d-1}$, T is a nonabelian simple group and $d > 1$;
5. almost simple: $T \leq G \leq \mathrm{Aut}\, T$, where T is a nonabelian simple group.

The imprimitive case has been studied at length in [2]. In this paper we examine the other cases in the hope of discovering interesting families of t-designs. First we note that if G is t-homogeneous on X, then, for every subset B of X of size at least t, the pair

(X,B^G) will be a block transitive t-design, and will be a nontrivial design as long as G is not $|B|$-homogeneous. Thus we shall always assume that G is not t-homogeneous. The paper [3] investigates block-transitive and flag-transitive t-designs with large t. (Recall that a <u>flag</u> in a design is an incident point-block pair.) It follows from a theorem of Ray–Chaudhuri and Wilson [6] that a block-transitive automorphism group of a t-design is $\lfloor t/2 \rfloor$-homogeneous on points, and a flag-transitive automorphism group of a t-design is $\lfloor (t+1)/2 \rfloor$-homogeneous on points. It is shown in [3] that there are no nontrivial block-transitive 8-designs and no nontrivial flag-transitive 7-designs. In this paper we shall concentrate on t-designs for small t (usually $t = 2$ or $t = 3$) and shall examine the possible automorphism groups type by type.

Further if (X, B^G) is a block-transitive t-design then also $(X, (X-B)^G)$ is a block-transitive t-design, so we may assume that $t < k \leq v/2$.

2. <u>The affine case</u>.

Let $G = AGL(d,p) < Sym(X)$ where $v = |X| = p^d$, p is prime and $d \geq 1$. Then G is 2-transitive, and, if $p = 2$, G is 3-transitive. Thus we shall look for 3-designs when p is odd and for 4-designs when $p = 2$. A search for block-transitive and flag-transitive 5-designs admitting $AGL(d,2)$ is described in [3].

Now let p be an odd prime and consider the case $d \geq 2$. Then G has 2 orbits on 3-element subsets of X, namely the sets Q_1 and Q_2 of collinear triples and non-collinear triples respectively. By [2, Proposition 1.3], for a k-element subset B of X, (X,B^G) is a 3-design if and only if $q_1/|Q_1| = q_2/|Q_2|$ where q_1, q_2 are the numbers of collinear and non-collinear triples in B respectively. Moreover $q_1 + q_2 = \binom{k}{3}$ so we have the following result.

<u>Lemma 2.1.</u> If $G = AGL(d,p) \leq Sym(X)$ with $d \geq 2$ and p an odd prime, and if B is a k-element subset of X, where $k \geq 3$, then the pair (X,B^G) is a block-transitive 3-design if and only if the number

q_1 of collinear triples in B is

$$\frac{k(k-1)(k-2)(p-2)}{6\ (p^d - 2)} .$$

It seems unlikely that a large family of 3-designs of this type will be found as the divisibility condition seems so difficult to satisfy. If u is a prime dividing $p^d - 2$ then u is odd and so u divides at most one of $k, k-1$ and $k-2$. If, in particular, $p^d - 2 = u^a$ then, when $k \le p^d/2$, u must be a divisor of $p-2$. From these observations it follows for example that when $p = 3$ we must have $d \ge 7$ and $3^7 - 2 = 5.19.23$. Is there a block-transitive $3-(3^7,k,\lambda)$ design of this type?

If $G = AGL(d,2)$ with $d \ge 3$ then G has 2 orbits on 4-element subsets of X, namely affine planes, and non-coplanar 4-sets. Applying [2, Proposition 1.3] we have

<u>Lemma 2.2.</u> If $G = AGL(d,2)$ with $d \ge 3$ and if $B \subseteq X$ with $|B| = k \ge 4$ then the pair (X, B^G) is a block-transitive 4-design if and only if the number q of affine planes in B is

$$\frac{k\ (k-1)\ (k-2)\ (k-3)}{24\ (2^d - 3)} .$$

This situation has been studied in more detail in [3] which looks at the problem of classifying all flag-transitive 5-designs. It is shown there that, for $G = AGL(d,2)$, (X, B^G) is a 4-design if and only if (X, B^G) is a 5-design, a very surprising result. From the divisibility condition above it follows that $d \ge 8$, and if $d = 8$ then the only integers k satisfying the divisibility condition are 23, 24, 25, 46, 47, 69, 209, 210, 232, 233. If the design is assumed to be flag-transitive then k must divide $|G|$ and so k is 24, 25, or 210. Moreover it is shown in [3] that there is indeed a flag-transitive $5-(2^8, 24, \lambda)$ design (where $\lambda = 2^{24}.3^2.5^2.7.31$) related to the extended Golay code, and there are no flag-transitive designs with $k = 25$ or $k = 210$.

3. The product action case.

Consider the wreath product $G = S_c \text{ wr } S_d$ in product action on X, that is $v = |X| = c^d$ and X is identified with Y^d where Y is a set of size c. Since we want G to be maximal in $\mathrm{Sym}(X)$ we have $c \geq 5$ and $d \geq 2$. Now G has d orbits on unordered pairs of points of X, namely $Q_1, \ldots, Q_d$, where $\{x, y\} \in Q_i$ if and only if x and y differ at exactly i entries, for $i = 1, \ldots, d$. The following criterion for a 2-design follows immediately from [2, Proposition 1.3].

Lemma 3.1. Let $G = S_c \text{ wr } S_d \leq \mathrm{Sym}(X)$, where $X = Y^d$, $|Y| = c$, and let $B \subseteq X$ be such that, for $1 \leq i \leq d$, q_i unordered pairs of points of B belong to Q_i, where $|B| = k$ and $\Sigma q_i = \binom{k}{2}$. Then (X, B^G) is a 2-design if and only if,

$$\frac{q_1}{\binom{d}{1}(c-1)} = \frac{q_2}{\binom{d}{2}(c-1)^2} = \cdots = \frac{q_d}{\binom{d}{d}(c-1)^d}.$$

In the case $d = 2$ this lemma leads to a simple construction principle. When $d = 2$, the points of X are ordered pairs of elements from the set Y of size c, and the k-set B can be interpreted as the edge set of a directed graph with vertex set Y. Note that loops are allowed. For an edge $e = (y, y')$ the first entry y will be called the <u>tail</u> and the second entry y' the <u>head</u> of e. The conditions given by Lemma 3.1 under which (X, B^G) is a 2-design reduce to just one equation:

$$q_1 = \frac{k(k-1)}{c+1}.$$

Thus we have the following:

Theorem 3.2. Let $D = (Y, B)$ be a directed graph with vertex set Y of size c and edge set $B \subseteq Y \times Y$ of size k. Then the set of all images of B under the group $G = \mathrm{Sym}(Y) \text{ wr } S_2$ is the set of blocks of a 2-design if and only if the number of unordered pairs of edges of B with a common head or a common tail is exactly $k(k-1)/(c+1)$. Moreover the design will be flag-transitive if and only if the automorphism group of the directed graph D is edge-transitive.

Construction 3.3. Let $D_0 = (X_0, B)$ be a directed graph with vertex set X_0 of size c_0, no isolated vertices and with k edges such that the number q of unordered pairs of edges of D_0 having a common head or a common tail is a divisor of $k(k-1)$. Then, provided $c = \frac{k(k-1)}{q} - 1 \geq c_0$, the digraph $D = (X, B)$ obtained from D_0 by adding $c - c_0$ isolated vertices gives rise to a 2-design as described in Theorem 3.2.

Example 3.4. Let $k = 2s \geq 6$ and let $D_0 = (\mathbb{Z}_s, B)$ be an "undirected" cycle of length s, that is $B = \{(i, i+1) \mid i \in \mathbb{Z}_s\} \cup \{(i+1, i) \mid i \in \mathbb{Z}_s\}$. Then the number of pairs of edges sharing a head or a tail is s which divides $k(k-1) = 2s(2s-1)$. Then adding $3(s-1)$ isolated vertices yields a flag-transitive $2-((2k-3)^2, k, \lambda)$ design for some λ.

It is difficult to obtain a general construction for large d as the number of restrictions on the parameters increases. However one necessary condition is the following.

Corollary 3.5. With the notation of Lemma 3.1, a necessary condition for (X, B^G) to be a 2-design is that $d\binom{k}{2}$ is divisible by $(c^d-1)/(c-1)$. (In fact $q_1 = d\binom{k}{2}/((c^d-1)/(c-1))$.)

Proof. By Lemma 3.1, $q_i = \binom{d}{i}(c-1)^{i-1}q_1 / d$ and

$$\binom{k}{2} = \sum_{i=1}^{d} q_1 = q_1 \left(\sum_{i=1}^{d} \binom{d}{i} (c-1)^i\right) / (c-1)d = q_1 (c^d - 1)/(c-1)d.$$

Thus $d\binom{k}{2}$ is divisible by $(c^d - 1)/(c-1)$.

It may be helpful to use the language of coding theory to describe the situation here. If the set Y is taken as the set $\mathbb{Z}_c$ of integers modulo c then $\{x, y\} \in Q_i$ if and only if $x - y$ has weight i, that is has exactly i nonzero entries. Thus B contains q_i unordered pairs $\{x, y\}$ with $x - y$ of weight i for $i = 1, \ldots, d$. Since G is transitive on X we may assume that $\underset{\sim}{0} = (0, \ldots, 0) \in B$. Then, if (X, B^G) is a flag-transitive 2-design, there are $2q_i/k$ pairs $\{\underline{0}, y\}$ in B with $y = y - \underline{0}$ of weight i, that is there are $2q_i/k$ elements of $B - \{\underline{0}\}$ of weight i.

<u>Theorem 3.6</u>. Let $G = S_c \text{ wr } S_d \le \mathrm{Sym}(X)$ where $X = \mathbb{Z}_c^d$, and let B be a k-element subset of X containing $\underline{0} = (0, \ldots, 0)$. Then (X, B^G) is a flag-transitive 2-design if and only if

(i) the setwise stabilizer G_B of B is transitive on B, and

(ii) for each $1 \le i \le d$ there are $2q_i/k$ elements of $B - \{\underline{0}\}$ of weight i, where $q_i = \binom{d}{i} (c-1)^{i-1} q_1 / d$ $= \binom{k}{2} \binom{d}{i} (c-1)^i / (c^d - 1)$.

<u>Proof</u>. If (X, B^G) is a flag-transitive 2-design then G_B is transitive on B, and, by Lemma 3.1 and Corollary 3.5, the parameters q_1 are as in (ii). So, by the discussion above (ii) is true. Conversely if (i) and (ii) are true then B contains q_i pairs in Q_i with q_i as in Lemma 3.1. Hence (X, B^G) is a 2-design, and, as G_B is transitive on B, (X, B^G) is a flag-transitive design.

The conditions for a flag-transitive 2-design in this case are very restrictive: by Corollary 3.5, $k - 1 = \left(\frac{2q_1}{k}\right) \cdot \left(\frac{c^d - 1}{c - 1}\right) \cdot \frac{1}{d}$ $\ge (c^d - 1)/ d(c - 1)$, that is the block size is very large.

<u>Question 3.7</u>. Are there any flag-transitive (or even block-transitive) $2-(c^d, k, \lambda)$ designs admitting $S_c \text{ wr } S_d$ with $d \ge 3$?

4. <u>The simple diagonal case</u>

Let $G = T^\ell.(\mathrm{Out}\, T \times S_\ell) \le \mathrm{Sym}(X)$ act on X in its diagonal action, where T is a nonabelian simple group and $\ell \ge 2$. Let $N = T^\ell < G$, and let $D = \{\underline{t} = (t, \ldots, t) \mid t \in T\}$ be the natural diagonal subgroup of N. Then X can be identified with the set of right cosets of D in N with N acting by right multiplication. If $\alpha = D$ is the trivial coset then $G_\alpha = \mathrm{Aut}\, T \times S_\ell$ and $G = NG_\alpha$. Elements of $\mathrm{Aut}\, T$ act on X by conjugation and elements of S_ℓ act by permuting the entries of coset representatives $\underline{x}$ of cosets $D\underline{x}$.

It is very unlikely that there will be any interesting 2-designs arising from this family of groups as G acting on pairs of points has many orbits in general. Perhaps it is worth saying a little about the simplest case, namely the case $\ell = 2$. Here each coset of D in N has a unique representative with first entry 1_T and so we may identify X with T. With this identification, $\alpha = 1_T$, and for $x \in X = T$, elements $(t_1, t_2) \in N$, $\sigma \in \mathrm{Aut}\, T \le G_\alpha$, and $\tau = (12) \in S_2 \le G_\alpha$ act as follows.

$$(t_1, t_2) : x \longrightarrow t_1^{-1} x t_2$$
$$\sigma : x \longrightarrow x^\sigma$$
$$\tau : x \longrightarrow x^{-1} .$$

The orbits of G on unordered pairs from X correspond to "fusion" classes of elements of T, where the fusion class $\mathcal{F}(x)$ of x is $\mathcal{F}(x) = \{(x)^{\epsilon\sigma} \mid \sigma \in \mathrm{Aut}\, T, \epsilon = \pm 1\}$: $\{x, y\}$ and $\{x', y'\}$ are in the same orbit on pairs if and only if $x^{-1}y$ and $(x')^{-1} y'$ are in the same fusion class. Let the fusion classes be $F_1 = \{1_T\}, \ldots, F_s$, let B be a k-element subset of T and let f_i be the number of unordered pairs of elements of B lying in F_i, for $i = 1, \ldots, s$. Then by [2, Proposition 1.3], (X, B^G) is a 2-design if and only if $f_i / \mid F_i \mid = E$ is independent of i (for $i = 1, \ldots, s$). Note that $\binom{k}{2} = \Sigma f_i = E\, \Sigma |F_i| = E\,(|T| - 1)$, so that k cannot be much smaller than $|T|^{1/2}$. Suppose now that G acts flag-transitively on (X,B^G). Then k divides $|G|$, and hence $(|T| - 1) / y$ divides $k - 1$ where y is the greatest common division of $|T| - 1$ and $|G|$. Since $k < v = |T|$ it follows that $y > 1$. Now $y = (|T| - 1 , |\mathrm{Out}\, T|)$ and it follows that T is a group of Lie type over a field of order p^a for some prime p and positive integer a, and y divides the odd part a' of a. Thus we have $k = 1 + z(|T| - 1)/y$ for some $1 \le z < y$. This means, on the one hand, that $k > |T|/a'$, and on the other hand that $(k, |T|)$ divides $(z - y , |T|)$, whence $(k,|G|)$ divides $2(z - y)^2 |\mathrm{Out}\, T|$. Since k divides $|G|$ it follows that $|T|/a' < k < 2(a')^2|\mathrm{Out}\, T|$. Thus $|T| < 2(a')^3|\mathrm{Out}\, T|$, and the only group satisfying this inequality is $T = PSL(2,8)$, but for this group $y = 1$. Thus G is never flag-transitive on (X, B^G).

Theorem 4.1. If $G = T^2 (\mathrm{Out}\ T \times S_2) \le \mathrm{Sym}(T)$ in simple diagonal action, where T is a nonabelian simple group, then G does not act flag-transitively on any nontrivial 2-design with point set T.

Question 4.2. Can $G = T^2 (\mathrm{Out}\ T \times S_2)$ act block-transitively on a nontrivial 2-design with point set T ?

5. The almost simple Case.

This case is the most difficult to discuss as the maximal almost simple subgroups of $\mathrm{Sym}\,(X)$ are only very loosely classified in [4]. There may be interesting classes of block-transitive 2-designs admitting primitive almost simple groups of small rank $\ell \ge 3$. For example in the rank 3 case we have:

Lemma 5.1. Let $G \le \mathrm{Sym}\,(X)$ be a primitive rank 3 group of degree v such that, for $x \in X$, G_x has a self-paired orbit $\Gamma(x)$ in $X - \{x\}$ of length m. Let B be a k-element subset of X and let q be the number of unordered pairs $\{x,y\}$ of points of B such that $y \in \Gamma(x)$ (or equivalently $x \in \Gamma(y)$). Then (X, B^G) is a block-transitive 2-design if and only if $q = \binom{k}{2} m / (v - 1)$.

In [2, Example 1.4] a construction of 2-designs was given based on the rank 3 groups $G = S_n$ acting on $v = \binom{n}{2}$ unordered pairs from a set Y of size n. In this case the set B can be interpreted as the edge set of a graph with vertex set Y having k edges. A 2-design was obtained if and only if the number of (unordered) pairs of edges of (Y, B) sharing a common vertex was $2k\,(k - 1) / (n + 1)$, and the design was flag-transitive if and only if the automorphism group of (Y, B) was edge-transitive.

Other classes of rank 3 groups may give similar constructions. For example the groups $G = P\Gamma L(n, q)$, $n \ge 4$, induce a primitive rank 3 action on the set of lines of the projective geometry $PG(n - 1, q)$.

Theorem 5.2. Let $G = P\Gamma L(n, q)$, $n \ge 4$, act on the set X of lines of $PG(n - 1, q)$, and let B be a k-element subset of X. Then (X, B^G) is a block-transitive 2-design if and only if the number of unordered pairs of intersecting lines in B is $\binom{k}{2}(q+1)^2(q-1) / (q^n + q^2 - q - 1)$.

Proof. Now $v = |X| = (q^n - 1)(q^{n-1} - 1) / (q^2 - 1)(q - 1)$ so $v - 1 = q(q^{n-2} - 1)(q^n + q^2 - q - 1) / (q^2 - 1)(q - 1)$. Also the number of lines intersecting a given line is $m = q(q^{n-2} - 1)(q + 1) / (q - 1)$. The result now follows from Lemma 5.1.

When considering primitive groups of rank greater than 3 the number of conditions to be satisfied increases and the problem of finding 2-designs becomes more difficult. We give just one example.

Theorem 5.3. Let $G = S_n$, the symmetric group on a set Y of size n and consider the primitive rank $s + 1$ action of G on the set X of $v = \binom{n}{s}$ s-element subsets of Y where $3 \le s \le n/2$. Let B be a k-element subset of X. Then (X, B^G) is a block-transitive 2-design if and only if, for each $i = 1, \ldots, s - 1$, the number q_i of unordered pairs of elements of B which intersect in exactly i elements of Y is

$$q_i = \frac{k(k-1)\binom{s}{i}\binom{n-s}{s-i}}{2\left(\binom{n}{s} - 1\right)}.$$

Proof The group G has s orbits $Q_0, \ldots, Q_{s-1}$ on unordered pairs of s-subsets of Y, namely Q_i consists of pairs which intersect in exactly i points of Y, for $0 \le i \le s - 1$. By [2, Proposition 1.3], (X, B^G) is a block-transitive 2-design if and only if $q_0/|Q_0| = \cdots = q_{s-1}/|Q_{s-1}| = x$ say. Then $\binom{k}{2} = \Sigma q_i = x\,\Sigma|Q_i| = x\binom{v}{2}$ and so these equations are equivalent to the equations $q_i = x|Q_i| = \binom{k}{2}|Q_i| / \binom{v}{2}$ for each $i = 1, \ldots, s - 1$, (since q_0 is determined by $\binom{k}{2} = \Sigma q_i$). This yields the result since $|Q_i| = v\binom{s}{i}\binom{n-s}{s-i}/2$ for $i = 0, 1, \ldots, s - 1$.

Example 5.4 Taking $s = 3$, we may interpret X as the set of triangles (cycles of length 3) of the complete graph with vertex set

Y, and we may interpret B as the set of triangles of a graph with vertex set Y having k triangles. Then, by Theorem 5.3, (X,B^G) is a 2–design if and only if the number q_2 of points of triangles in B sharing an edge is $3k(k-1)(n-3)/2(v-1) = 9k(k-1)/(n^2+2)$ and the number q_1 of pairs of triangles in B with a single vertex in common is $3k(k-1)\binom{n-3}{2}/2(v-1) = 9k(k-1)(n-4)/2(n^2+2)$.

On the other hand if G is 2–transitive then we should be looking for t–designs with $t \geq 3$. We do this for the projective linear groups below.

<u>Theorem 5.5</u> Consider $G = P\Gamma L(n,q)$, $n \geq 3$, acting on the set X of $v = (q^n-1)/(q-1)$ points of the projective geometry $PG(n-1,q)$, and let B be a k–elelment subset of X. Then (X,B^G) is a block–transitive 3–design if and only if the number of (unordered) collinear triples of points in B is

$$k(k-1)(k-2)(q-1)^2/6(q^n-2q+1)$$
$$= k(k-1)(k-2)(q-1)/6(v-2) .$$

<u>Proof</u> The group G has two orbits on unordered triples of distinct points, namely on collinear triples and non–collinear triples and there are $m = v(v-1)(q-1)/6$ collinear triples. By [2, Proposition 1.3] the condition for a 3–design is that the number of collinear triples in B is $\binom{k}{3}m/\binom{v}{3}$.

<u>Example 5.6</u> If $G = PGL(3,7)$ then the number of collinear triples in B is $c = k(k-1)(k-2)/55$ and so k is 11, 12, 22, 35, 45, or 46. An example with $k = 11$ can be constructed as follows: Note that B must contain $c = 18$ collinear triples in this case. Let 0 be an oval in $PG(2,7)$, that is a set of 8 points with no three collinear. Let $\alpha_1,\alpha_2 \in 0$, let ℓ be the line through α_1 and α_2, and let $\alpha_3,\alpha_4,\alpha_5,\alpha_6$ be four distinct points on $\ell - \{\alpha_1,\alpha_2\}$. Set $B = \{\alpha_3,\alpha_4,\alpha_5,\alpha_6\} \cup (0 - \{\alpha_1\})$. Then $|B| = 11$. The only collinear triples in B containing at least two points of $B - 0$ are triples

from $\{\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\} \subseteq \ell$ and there are 10 of these. The only other collinear triples in B contain one point of $B - 0$ and two points of 0 (that is they are on secant lines to 0 different from ℓ and passing through one of $\alpha_3,\alpha_4,\alpha_5,\alpha_6$), and there are 8 of these, two containing each of $\alpha_3,\alpha_4,\alpha_5$ and α_6. Thus (X,B^G) is a block-transitive 3-(57,11,λ) design, for some λ, admitting G.

Similarly there is an example with $k = 12$ and $c = 24$ constructed as follows. Let β be a point not on 0 or ℓ such that the lines through β and α_1 and through β and α_2 are both secant lines to 0 (see Figure 1). Choose α_3 and α_4 on ℓ such that the lines through β and α_3 and through β and α_4 are both tangent lines to 0. Finally choose α_5 such that the line through β and α_5 is a secant line to 0 and choose α_6 such that the line through β and α_6 is an external line to 0.

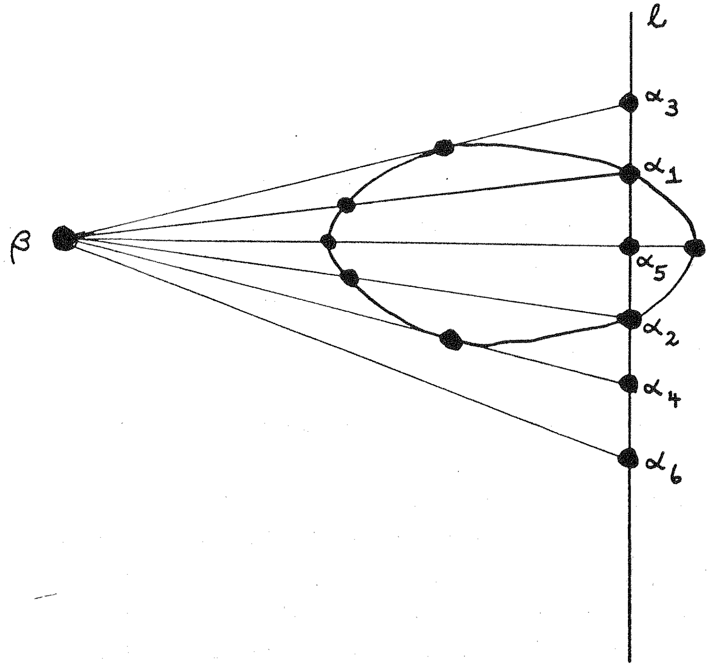

Figure 1

Let $B = \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta\} \cup (0 - \{\alpha_1\})$. Then $|B| = 12$. There are 10 collinear triples in B containing 3 points of ℓ . There are 7 collinear triples in B containing β , namely each of $\alpha_2, \alpha_3, \alpha_4$ lies in one such triple and there are 4 triples in B on the line through β and α_5 . The remaining triples lie on secant lines to 0 not on β , and contain two points of $0 - \ell$ and one pointof $\ell - 0$: each of α_3, α_4 and α_6 lie on two such triples, and α_5 lies on one such triple. Thus B contains 24 collinear triples and so (X, B^G) is a block-transitive $3 - (57,12,\lambda)$ design, for some λ , admitting G .

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