DEFICIENCIES AND VERTEX CLIQUE COVERING NUMBERS OF A FAMILY OF TREES

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ABSTRACT: Let G be a simple graph on n vertices having a maximum matching M. The **deficiency** def(G) of G is the number of M-unsaturated vertices in G. The **vertex clique covering number** vcc(G) of G is the smallest number of cliques (complete subgraphs) needed to cover the vertex set of G. In this paper we determine def(G) and vcc(G) for the case when G is a tree with each vertex having degree 1 or k.

1. INTRODUCTION

All graphs considered in this paper are finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [1]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices and $\varepsilon(G)$ edges.

A **matching** M in G is a subset of E(G) in which no two edges have a vertex in common. M is a **maximum matching** if $|M| \geq |M'|$ for any other matching M′ of G. A vertex v is **saturated** by M if some edge of M is incident with v; otherwise v is said to be unsaturated. The **deficiency** def(G) of G is the number of unsaturated vertices by any maximum matching M of G. If def(G) = 0, then, of course, G has a perfect matching. For a maximum matching M we have $|M| = \frac{1}{2}(n - \text{def}(G))$. Many problems concerning matchings in graphs have been investigated in the literature - see, for example, Lovasz and Plummer [7]. In this paper we consider the problem of determining def(G); results for a family of trees are obtained.

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A **clique** of G is a complete subgraph of G. The **clique covering number (clique partion number)** cc(G) (cp(G)) of G is the smallest number of cliques (edge-disjoint cliques) needed to cover the edge set of G. The **vertex clique covering number** vcc(G) of G is the minimum number of cliques needed to cover the vertex set of G. Many authors have studied the functions cc(G) and cp(G) - see for example Caccetta and Pullman [2], Pullman [10] and Ma et.al. [8]. In this paper we investigate the function vcc(G). Observe that $vcc(G) \le |M| + def(G) = \frac{1}{2}(\nu + def(G))$, with equality holding when G is triangle free. So the functions def(G) and vcc(G) are related. Further the vcc(G) is the same as the chromatic number $\chi(\bar{G})$ of $\bar{G}$. The chromatic number of regular graphs has been studied by Caccetta and Pullman [3-4].

The results we present are for the case when G is a tree in which each vertex has degree 1 or k. We let $\tau(n;1,k)$ denote the class of trees on n vertices in which each vertex has degree 1 or k, $k \ge 2$.

2. RESULTS

We begin by making some simple observations concerning def(G). The definition implies that $def(G) \equiv \nu (\text{mod } 2)$. If $\varepsilon(G) > 0$, then $0 \le def(G) \le \nu - 2$. Consider the tree T on n vertices drawn in Figure 1 below, where $d \equiv n(\text{mod } 2)$.

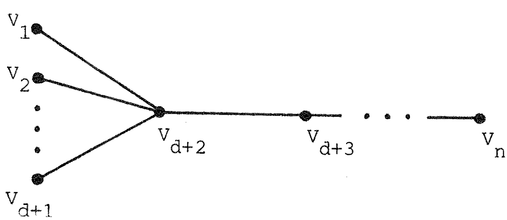

Figure 1

Clearly def(T) = d. Consequently, if D(n) denotes the set of possible values of def(G) as G ranges over the class of simple non-empty graphs on n vertices, then

$$D(n) = \{d:\ 0 \le d \le n-2,\ d \equiv n (\mathrm{mod}\ 2)\} .$$

Thus we need to look at restricted classes of graphs to obtain more interesting results on def(G). We consider the case when G is a tree. In view of the graph displayed in Figure 1 we need to add some further restrictions. We now consider the class $\tau(n;1,k)$.

Lemma 1. Let $T \in \tau(n;1,k)$ be a graph with s vertices of degree k. Then

$$\mathrm{def}(T) \le (k - 3)s + 2 + 2\lfloor (s - 1)/k \rfloor . \qquad (1)$$

Proof: Simple counting yields

$$s = (n - 2)/(k - 1) . \qquad (2)$$

We prove the lemma using induction on s. When $s = 0$, $T = K_2$ and $\mathrm{def}(T) = 0$. When $s = 1$, then T is a star with $n = k + 1$ and hence $\mathrm{def}(T) = k - 1$. Thus the result is true for $s = 0$ and $s = 1$. Assume it is true for all $1 \le s \le m$ and let T be a tree with $s = m + 1$.

Equation (2) implies that $n = (k - 1)(m + 1) + 2$. If T contains a vertex, u say, of degree k that is joined to a vertex of degree 1, then every maximum matching saturates u. Furthermore, there exists a maximum matching M which contains an edge uv with $d_T(v) = 1$. So let us assume that M is a maximum matching in T which contains such edges. Suppose T contains s′ vertices of degree k adjacent to vertices of degree 1, then

$$s' \ge \frac{n - m - 1}{k - 1}$$

$$= m + 1 - \frac{m - 1}{k - 1} .$$

If T has no M-unsaturated vertices of degree k, then

$$\begin{aligned} \operatorname{def}(T) &= (n - m - 1) - s' \\ &\le (k - 2)(m + 1) + 2 - (m + 1) + \frac{m - 1}{k - 1} . \end{aligned}$$

Hence

$$\operatorname{def}(T) \le (k - 3)(m + 1) + 2 + \left\lfloor \frac{m - 1}{k - 1} \right\rfloor .$$

Now if $k > m$, then $\lfloor (m - 1)/(k - 1) \rfloor = \lfloor m/k \rfloor = 0$ and hence (1) holds. So we can suppose that $2 \le k \le m$.

Then $(k - 2)(k - m) \le 0$ and so $(m - 1)k \le (k - 1)(2m - k)$.

Hence

$$\begin{aligned} \left\lfloor \frac{m - 1}{k - 1} \right\rfloor &\le \left\lfloor \frac{2m - k}{k} \right\rfloor \\ &\le 2 \left\lfloor \frac{m}{k} \right\rfloor . \end{aligned}$$

Thus

$$\operatorname{def}(T) \le (k - 3)(m + 1) + 2 + 2\left\lfloor \frac{m}{k} \right\rfloor ,$$

proving that (1) holds for $s = m + 1$ when T has no M-unsaturated vertices.

Now suppose that T has M-unsaturated vertices of degree k. Let u be such a vertex and let $N_T(u) = \{v_1, v_2, \ldots, v_k\}$ denote the neighbours of u. Form the graph T′ from T as follows. Delete u and add k new vertices $u_1, u_2, \ldots, u_k$ and k new edges $v_i u_i$, $1 \le i \le k$ (see Figure 2).

Then T' consists of k trees $T_1, T_2, \ldots, T_k$ with $T_i \in \tau(n_i;1,k)$. Suppose T_i has s_i vertices of degree k. Then $s_i \le m$ for all i and $\sum_{i=1}^{k} s_i = m$. Now, by our induction hypothesis,

$$\mathrm{def}(T_i) \le (k-3)\,s_i + 2 + 2\left\lfloor \frac{s_i - 1}{k} \right\rfloor .$$

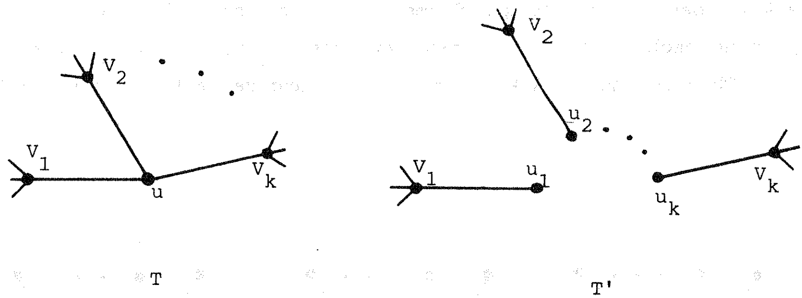

Figure 2

For each i there exists a maximum matching in T_i that does not saturate u_i. We have

$$\begin{aligned}
\mathrm{def}(T) &= \sum_{i=1}^{k} \mathrm{def}(T_i) - k + 1 \\
&\le (k-3) \sum_{i=1}^{k} s_i + 2k + 2 \sum_{i=1}^{k} \left\lfloor \frac{s_i - 1}{k} \right\rfloor - k + 1 \\
&= (k-3)m + k + 1 + 2 \sum_{i=1}^{k} \left\lfloor \frac{s_i - 1}{k} \right\rfloor \\
&\le (k-3)m + k + 1 + 2 \left\lfloor \sum_{i=1}^{k} \frac{s_i - 1}{k} \right\rfloor \\
&= (k-3)m + k + 1 + 2 \left\lfloor \frac{m-k}{k} \right\rfloor
\end{aligned}$$

$$= (k - 3)(m + 1) + 2 + 2\lfloor m/k \rfloor .$$

Thus (1) holds for $s = m + 1$. This completes the proof of (1).

□

We now demonstrate that the bound given in Lemma 1 is sharp. This is obviously the case for $k = 2$, so we suppose that $k \geq 3$. Let $A(k,t)$ denote the graph formed from the path $P = v_1, v_2, \ldots, v_t$ by joining each v_i to $k - 2$ new vertices $v_{i1}, v_{i2}, \ldots, v_{i,k-2}$ (see Figure 3). Observe that $\nu(A(k,t)) = t(k - 1)$ and $\mathrm{def}(A(k,t)) = t(k - 3)$.

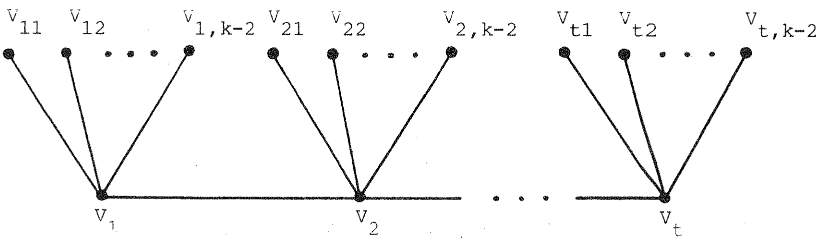

Figure 3 A(k,t)

Consider the graph $A(k,2)$. We form the graph $B(k)$ by adding $(k - 2)$ disjoint copies of $\bar{K}_{k-1}$ and joining v_{2i}, $1 \leq i \leq k-2$, to all the vertices of the i^{th} copy of $\bar{K}_{k-1}$ (see Figure 4). Observe that $B(k)$ is a tree with $k(k - 1)$ vertices and deficiency $(k - 1)(k - 2)$. We will now construct, using $A(k,t)$ and $B(k)$ as building blocks, a graph $T \in \tau(n;1,k)$ with $\mathrm{def}(T)$ equal to the right hand side of (1).

Let $s - 1 = kp + r$, $0 \leq r \leq k - 1$ and $p \geq 0$. If $p = 0$, we can take our T as the graph formed from $A(k,r+1)$ by joining v_1 and v_{r+1} to two new vertices u_1 and u_2, respectively. When $p > 0$ we form our T as follows.

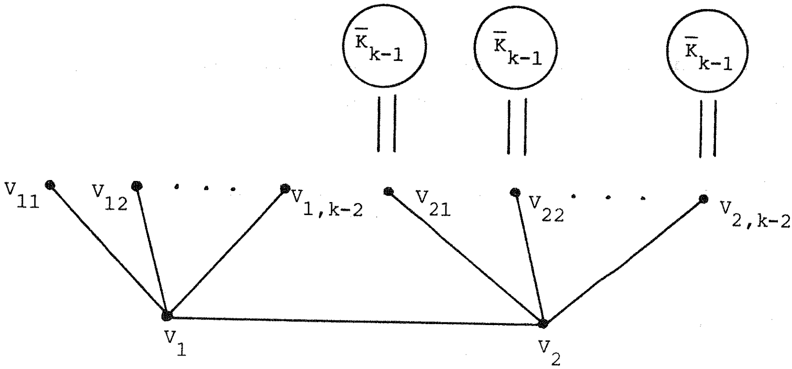

Figure 4 B(k)

Take p copies of $B_1(k), B_2(k), \ldots, B_p(k)$ and $A(k,r+1)$. Identify the "v_1, v_2 vertices" of $B_i(k)$, $1 \le i \le p$, by $v_1^{(i)}$ and $v_2^{(i)}$, respectively. We join the vertex $v_2^{(i)}$ to $v_1^{(i+1)}$ for each $1 \le i \le p-1$, join $v_1^{(1)}$ to a new vertex u_1, join $v_2^{(p)}$ to the vertex v_1 of $A(k,r+1)$ and join the vertex v_{r+1} of $A(k,r+1)$ to a new vertex u_2. Call the resulting graph T. Observe that every vertex of T has degree 1 or k and

$$\begin{aligned} \nu(T) &= pk(k-1) + (r+1)(k-1) + 2 \\ &= s(k-1) + 2 . \end{aligned}$$

Thus $T \in \tau(n;1,k)$ with $n = s(k-1) + 2$. Now every maximum matching of T saturates the vertices $v_1^{(i)}$, $1 \le i \le p$, $v_1, v_2, \ldots v_{r+1}$. Also, every maximum matching of $B_i(k)$ saturates $v_1^{(i)}$, $1 \le i \le p$. Consequently

$$\begin{aligned} \mathrm{def}(t) &= \sum_{i=1}^{p} \mathrm{def}(B_i(k)) + \mathrm{def}(A(k,r+1)) + 2 \\ &= p(k-1)(k-2) + (r+1)(k-3) + 2 \end{aligned}$$

$$= (k - 3)s + 2 + 2 \lfloor (s - 1)/k \rfloor .$$

This establishes that the upper bound given by Lemma 1 is sharp. We now turn our attention to the lower bound.

Lemma 2. Let $T \in \tau(n;1,k)$ be a graph with s vertices of degree k. Then

(a) $\text{def}(T) = 0$, if $s = 0$

(b) $\text{def}(T) \geq (k - 3)s + 2$, if $s \geq 1$

and this bound is sharp for $k \geq 3$.

Proof: The lemma will be proved by using induction on s. The result is true for $s = 0$ and $s = 1$. Assume it is true for all s, $1 \leq s \leq m$, and let T be a tree with $s = m + 1$. Let u be a vertex of degree k in T. As in the proof of Lemma 1 we form T' from T by deleting u and adding k new vertices $u_1, u_2, \ldots u_k$ and k new edges $u_i v_i$, $1 \leq i \leq k$, (see Figure 2). Then T' is a forest consisting of k components $T_1, T_2, \ldots, T_k$ with $u_i \in T_i$. Let $n_i = |V(T_i)|$. Clearly $T_i \in \tau(n_i;1,k)$ and $\sum_{i=1}^{k} n_i = n + k - 1$.

Suppose T_i has s_i vertices of degree k. Then $s_i \leq m$ and $\sum_{i=1}^{k} s_i = m$. By our induction hypothesis we have

$$\text{def}(T_i) \geq (k - 3)s_i + 2 \qquad \text{when } s_i > 0.$$

If $s_i = 0$, then T_i is necessarily an edge and hence $\text{def}(T_i) = 0$. On the other hand, if $s_i > 0$ then v_i is saturated by every maximum matching of T_i; u_i may or may not be saturated. Suppose T' has p components with no vertices of degree k. Without loss of generality we may take these components as $T_1, T_2, \ldots, T_p$. We have

$$\begin{aligned} def(T) &\geq def(T') - (k - p) + p - 1 \\ &= \sum_{i=p+1}^{k} def(T_i) - (k - p) + p - 1 \\ &\geq \sum_{i=p+1}^{k} [(k - 3)s_i + 2] - k + 2p - 1 \\ &= (k - 3)m + 2(k - p) - k + 2p - 1 \\ &= (k - 3)(m + 1) + 2 . \end{aligned}$$

This completes the proof of the inequality in (b).

That the bound is sharp follows from the graph T obtained from A(k,s) by adding two new vertices u_1 and u_2 and the edges u_1v_1 and v_su_2.

□

Let

$$D(n;1,k) = \{def(T): T \in \tau(n;1,k)\} .$$

When k = 2, $\tau(n;1,k)$ consists of just a path of length n - 1 and hence $D(n;1,k) = \{0\}$ if n is even and $D(n;1,k) = \{1\}$, if n is odd. For $k \geq 3$ we have the following result.

Theorem 1. For integers $k \geq 3$ and $n \geq 4$,

(a) $D(n;1,k) = \phi$, if $n \not\equiv 2(\text{mod}(k - 1))$, and

(b) $D(n;1,k) = \{d\colon \frac{(k-3)n+4}{k-1} \le d \le \frac{(k-3)n+4}{k-1} + 2 \left\lfloor\frac{n-k-1}{k(k-1)}\right\rfloor,$

$d \equiv n \pmod 2\}, \quad \text{if } n \equiv 2(\text{mod}(k-1)).$

Proof: Let $T \in \tau(n;1,k)$ be a graph with s vertices of degree k. Equation (2) implies that $n \equiv 2(\text{mod}(k - 1))$, proving (a). Now suppose that $n = s(k - 1) + 2$ and let $s - 1 = kp + r$, where $0 \le r \le k - 1$ and $p \ge 0$. Then $n = (k - 1)(kp + r + 1) + 2$. Hence

$$\frac{(k-3)n+4}{k-1} = (k-3)(kp+r+1) + 2,$$

and

$$\left\lfloor\frac{n-k-1}{k(k-1)}\right\rfloor = \left\lfloor\frac{kp+r}{k}\right\rfloor = p.$$

Thus we must prove that

$$D(n;1,k) = \{d\colon (k-3)(kp+r+1) + 2 \le d \le (k-3)(kp+r+1) + 2 + 2p, \quad d \equiv n \pmod 2\}. \tag{3}$$

We do this by construction.

Let

$$d = (k-3)(kp+r+1) + 2 + 2q, \quad 0 \le q \le p.$$

We have already given constructions for the lower bound $q = 0$ and the upper bound $q = p$. So suppose $1 \le q \le p - 1$. Using the building blocks $B(k)$ and $A(k,t)$ defined earlier we shall form a graph $T_q \in \tau(n;1,k)$ having deficiency d.

Take q copies $B_1(k), B_2(k), \ldots, B_q(k)$ of $B(k)$ and the graph $A(k, k(p - q) + r + 1)$. As earlier we identify the vertices v_1 and v_2 of $B_i(k)$, $1 \le i \le q$, by $v_1^{(i)}$ and $v_2^{(i)}$, respectively. We join: $v_2^{(i)}$ to $v_1^{(i+1)}$ for each $1 \le i \le q - 1$; $v_1^{(1)}$ to a new vertex u_1; $V_2^{(q)}$ to the vertex v_1 of $A(k, k(p - q) + r + 1)$; and the vertex $v_{k(p-q)+v+1}$ of $A(k, k(p - q) + r + 1)$ to a new vertex u_2. Call the resulting graph T_q. Then $T_q \in \tau(n; 1, k)$ as every vertex of T_q has degree 1 or k and

$$
\begin{aligned}
\nu(T_q) &= q\, k(k - 1) + (k(p - q) + r + 1)(k - 1) + 2 \\
&= (k - 1)(pk + r + 1) + 2 \\
&= n\ .
\end{aligned}
$$

Further, using the arguments following Lemma 1, we have

$$
\begin{aligned}
\mathrm{def}(T_q) &= \sum_{i=1}^{q} \mathrm{def}(B_i(k)) + \mathrm{def}(A(k, k(p - q) + r + 1)) + 2 \\
&= q(k - 1)(k - 2) + (k(p - q) + r + 1)(k - 3) + 2 \\
&= (k - 3)(kp + r + 1) + 2 + 2q\ .
\end{aligned}
$$

Hence T_q has deficiency d as required. The theorem now follows from Lemmas 1 and 2. □

Let

$$V(n; 1, k) = \{\mathrm{vcc}(T) : T \in \tau(n; 1, k)\}\ .$$

Since $\tau(n;1,2)$ consists of just a path of length $n - 1$, we have $V(n;1,2) = \{\lceil \frac{n}{2} \rceil\}$. For $k \geq 3$, Theorem 1 and the fact that $vcc(T) = \frac{1}{2}(n + def(T))$ yields:

Theorem 2. For integers $k \geq 3$ and $n \geq 4$,

(a) $V(n;1,k) = \phi$, if $n \not\equiv 2 \pmod{k-1}$,

(b) $V(n;1,k) = \{x : \frac{(k-2)n+2}{k-1} \leq x \leq \frac{(k-2)n+2}{k-1} + \lfloor \frac{n-k-1}{k(k-1)} \rfloor ,$
$x \in \mathbb{N} \}$, if $n \equiv 2 \pmod{k-1}$.

□

The **edge covering number** $\rho(G)$ of G is the smallest number of edges needed to cover the vertex set of G. Let $\delta(G)$ denote the minimum degree of vertices of G. In 1959, Gallai proved (see p102 of [1]) the following result.

Theorem (Gallai) Let M be any maximum matching in G, where $\delta(G) > 0$. Then

$$\rho(G) + |M| = \nu(G) .$$

□

We have the following corollary.

Corollary. If G is triangle free and $\delta(G) > 0$, then

$$\rho(G) = vcc(G) .$$

□

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