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Generating functions for descents over permutations which avoid sets of consecutive patterns.

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Abstract

We extend the reciprocity method of Jones and Remmel ([*Discrete Math.* 313 (2013), 2712–2729] and [*Pure Math. Appl.* 24 (2013), 151–178]) to study generating functions of the form

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\Gamma)} x^{\mathrm{LRmin}(\sigma)}y^{1+\mathrm{des}(\sigma)}$$

where Γ is a set of permutations which start with 1 and have at most one descent, $\mathcal{NM}_n(\Gamma)$ is the set of permutations σ in the symmetric group $\mathfrak{S}_n$ which have no Γ-matches, $\mathrm{des}(\sigma)$ is the number of descents of σ and $\mathrm{LRmin}(\sigma)$ is the number of left-to-right minima of σ. We show that this generating function is of the form $\left(\frac{1}{U_\Gamma(t,y)}\right)^x$ where $U_\Gamma(t,y) = \sum_{n\geq 0} U_{\Gamma,n}(y)\frac{t^n}{n!}$ and the coefficients $U_{\Gamma,n}(y)$ satisfy some simple recursions in the case where Γ equals $\{1324, 123\}$, $\{1324\cdots p, 12\cdots(p-1)\}$ and $p\geq 5$, or Γ is the set of permutations $\sigma=\sigma_1\cdots\sigma_n$ of length $n=k_1+k_2$ where $k_1,k_2\geq 2$, $\sigma_1=1$, $\sigma_{k_1+1}=2$, and $\mathrm{des}(\sigma)=1$.

1 Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $\{1,\ldots,n\}$. If $\sigma = \sigma_1\cdots\sigma_n\in\mathfrak{S}_n$, we say that i is a descent of σ if $\sigma_i>\sigma_{i+1}$ and σ_j is a left-to-right minimum of σ if $\sigma_j<\sigma_i$ for all $i<j$. We let $\mathrm{des}(\sigma)$ be the number of descents of σ and $\mathrm{LRmin}(\sigma)$ be the number of left-to-right minima of σ. Given a sequence $\alpha=\alpha_1\cdots\alpha_n$ of distinct integers, the *reduction* of α, $\mathrm{red}(\alpha)$, is the permutation in $\mathfrak{S}_n$ found by replacing the i^{th} smallest integer that appears in α by i. For example, if $\alpha = 9\ 2\ 7\ 4\ 5$, then $\mathrm{red}(\alpha)=51423$. Let Γ be a set of permutations. We say that a permutation $\sigma=\sigma_1\cdots\sigma_n\in\mathfrak{S}_n$ has a Γ-match starting at position i if there is a $j\geq 1$ such that $\mathrm{red}(\sigma_i\sigma_{i+1}\cdots\sigma_{i+j})\in\Gamma$. We let $\Gamma\text{-mch}(\sigma)$ denote the number

of Γ-matches in σ. We let $\mathcal{NM}_n(\Gamma)$ be the set of permutations σ in the symmetric group $\mathfrak{S}_n$ such that Γ-mch$(\sigma) = 0$.

The main goal of this paper is to study the generating function

$$\mathrm{NM}_\Gamma(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{NM}_n(\Gamma)} x^{\mathrm{LRmin}(\sigma)} y^{1+\mathrm{des}(\sigma)}$$

in the case where Γ is a set of permutations such that for each $\alpha \in \Gamma$, α starts with 1 and $des(\alpha) \leq 1$. In the special case where Γ consists of a single permutation τ, we will denote $\mathrm{NM}_\Gamma(t,x,y)$ simply as $\mathrm{NM}\tau(t,x,y)$. Jones and Remmel [11] showed that if every permutation in Γ starts with 1, then we can write $\mathrm{NM}_\Gamma(t,x,y)$ in the form $\left(\frac{1}{U_\Gamma(t,y)}\right)^x$ where

$$U_\Gamma(t,y) = \sum_{n\geq 0} U_{\Gamma,n}(y)\frac{t^n}{n!}.$$

There is a considerable literature on the generating function $\mathrm{NM}_\Gamma(t,1,1)$ of permutations that consecutively avoid a pattern or set of patterns. See for example,[1–5, 7–10, 15–17]. For the most part, these papers do not consider generating functions of the form $\mathrm{NM}\tau(t,1,y)$ or $\mathrm{NM}\tau(t,x,y)$. An exception is the work on enumeration schemes of Baxter [2, 3], who gave general methods to enumerate patterns avoiding vincular patterns according to various permutations statistics. Our approach is to use the reciprocity method of Jones and Remmel.

Jones and Remmel [12–14] developed what they called the reciprocity method to compute the generating function $\mathrm{NM}\tau(t,x,y)$ for certain families of permutations τ such that τ starts with 1 and $\mathrm{des}(\tau) = 1$. The basic idea of their approach is as follows. First one writes

$$U_\tau(t,y) = \frac{1}{1 + \sum_{n\geq 1} \mathrm{NM}_{\tau,n}(1,y)\frac{t^n}{n!}}. \tag{1}$$

One can then use the homomorphism method to give a combinatorial interpretation of the right-hand side of (1) which can be used to find simple recursions for the coefficients $U_{\tau,n}(y)$. The homomorphism method derives generating functions for various permutation statistics by applying a ring homomorphism defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ to simple symmetric function identities such as

$$H(t) = 1/E(-t),$$

where $H(t)$ and $E(t)$ are the generating functions for the homogeneous and elementary symmetric functions, respectively:

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1} \frac{1}{1-x_i t}, \qquad E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1} 1 + x_i t. \tag{2}$$

In their case, Jones and Remmel defined a homomorphism θ on Λ by setting

$$\theta(e_n) = \frac{(-1)^n}{n!}\mathrm{NM}_{\tau,n}(1,y).$$

Then
$$\theta(E(-t)) = \sum_{n\geq 0} \mathrm{NM}_{\tau,n}(1,y)\frac{t^n}{n!} = \frac{1}{U_\tau(t,y)}.$$
Hence
$$U_\tau(t,y) = \frac{1}{\theta(E(-t))} = \theta(H(t)),$$
which implies that
$$n!\theta(h_n) = U_{\tau,n}(y).$$
Thus if we can compute $n!\theta(h_n)$ for all $n \geq 1$, then we can compute the polynomials $U_{\tau,n}(y)$ and the generating function $U_\tau(t,y)$, which in turn allows us to compute the generating function $\mathrm{NM}_\tau(t,x,y)$. Jones and Remmel [13, 14] showed that one can interpret $n!\theta(h_n)$ as a certain signed sum of weights of filled labeled brick tabloids when τ starts with 1 and $\mathrm{des}(\tau) = 1$. Then they showed how such a combinatorial interpretation allowed them to prove that for certain families of such permutations τ, the $U_{\tau,n}(y)$'s satisfied certain simple recursions.

The main purpose of this paper is to extend the methods of Jones and Remmel [13, 14] so that one can compute $U_{\Gamma,n}(y)$. In our case we assume that if $\tau \in \Gamma$, then τ starts with 1 and $\mathrm{des}(\tau) \leq 1$. One of the most interesting cases from our point of view is the case when Γ contains an identity permutation $12\cdots(k+1)$ where $k \geq 2$. In such a case, the underlying set of weighted filled labeled brick tabloids which we use to interpret $U_{\Gamma,n}(y)$ has the property that all the bricks have size less than or equal to k. This results in a significant difference between the recursions satisfied by $U_{\tau,n}(y)$ and the recursions satisfied by $U_{\{\tau,12\cdots(k+1)\},n}(y)$.

For example, in [13], Jones and Remmel studied the generating functions $\mathrm{NM}_\tau(t, x,y)$ for permutations τ of the form $\tau = 1324\cdots p$ where $p \geq 4$. That is, τ arises from the identity permutation by transposing 2 and 3. Using the reciprocity method, they proved that $U_{1324,1}(y) = -y$ and for $n \geq 2$,
$$U_{1324,n}(y) = (1-y)U_{1324,n-1}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} C_{k-1} U_{1324,n-2k+1}(y)$$
where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number. They also proved that for any $p \geq 5$, $U_{1324\cdots p,n}(y) = -y$ and for $n \geq 2$,
$$U_{1324\cdots p,n}(y) = (1-y)U_{1324\cdots p,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{p-2} \rfloor +1} (-y)^{k-1} U_{1324\cdots p,n-((k-1)(p-2)+1)}(y).$$

We will prove the following two theorems.

Theorem 1. *Let* $\Gamma = \{1324, 123\}$. *Then*
$$NM_\Gamma(t,x,y) = \left(\frac{1}{U_\Gamma(t,y)}\right)^x \text{ where } U_\Gamma(t,y) = 1 + \sum_{n\geq 1} U_{\Gamma,n}(y)\frac{t^n}{n!},$$

$U_{\Gamma,1}(y) = -y$, *and for* $n \geq 2$,

$$U_{\Gamma,n}(y) = -yU_{\Gamma,n-1}(y) - yU_{\Gamma,n-2}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^k C_{k-1} U_{\Gamma,n-2k}(y).$$

Theorem 2. *Let* $\Gamma = \{1324\ldots p, 123\ldots p-1\}$ *where* $p \geq 5$. *Then*

$$NM_\Gamma(t,x,y) = \left(\frac{1}{U_\Gamma(t,y)}\right)^x \text{ where } U_\Gamma(t,y) = 1 + \sum_{n\geq 1} U_{\Gamma,n}(y)\frac{t^n}{n!},$$

$U_{\Gamma,1}(y) = -y$, *and for* $n \geq 2$,

$$U_{\Gamma,n}(y) = \sum_{k=1}^{p-2}(-y)U_{\Gamma,n-k}(y) + \sum_{k=1}^{p-2}\sum_{m=2}^{\lfloor \frac{n-k}{p-2} \rfloor} (-y)^m U_{\Gamma,n-k-(m-1)(p-2)}(y).$$

Note that both Theorems 1 and 2 show that the reciprocity method applies even in cases where Γ is a family that contains permutations of different lengths. In the case of Theorem 1, the polynomials $U_{\{1324,123\},n}(-y)$ are the polynomials in the sequences A039598 and A039599 in On-line Encyclopedia of Integer Sequences [18] up to a power of y. The polynomials in sequences A039598 and A039599 are related to the expansions of the powers of x in terms of the Chebyshev polynomials of the second kind. We will give a bijection between our combinatorial interpretation of $U_{\{1324,123\},2n}(-y)$ and one of the known combinatorial interpretations for A039599, and a bijection between our combinatorial interpretation of $U_{\{1324,123\},2n+1}(-y)$ and one of the known combinatorial interpretations for A039598. This will allow us to give closed expressions for the polynomials $U_{\{1324,123\},n}(y)$. That is, we will prove that for all $n \geq 0$,

$$\begin{aligned} U_{\{1324,123\},2n}(y) &= \sum_{k=0}^{n} \frac{(2k+1)\binom{2n}{n-k}}{n+k+1}(-y)^{n+k+1} \text{ and} \\ U_{\{1324,123\},2n+1}(y) &= \sum_{k=0}^{n} \frac{2(k+1)\binom{2n+1}{n-k}}{n+k+2}(-y)^{n+k}. \end{aligned}$$

Another example is the following. Let $k_1, k_2 \geq 2$ and $p = k_1 + k_2$. We consider the family of permutations Γ_{k_1,k_2} in $\mathfrak{S}_p$ defined as

$$\Gamma_{k_1,k_2} = \{\sigma \in \mathfrak{S}_p : \sigma_1 = 1, \sigma_{k_1+1} = 2, \sigma_1 < \sigma_2 < \cdots < \sigma_{k_1} \ \& \ \sigma_{k_1+1} < \sigma_{k_1+2} < \cdots < \sigma_p\}.$$

That is, Γ_{k_1,k_2} consists of all permutations σ of length p where 1 is in position 1, 2 is in position k_1+1, and σ consists of two increasing sequences, one starting at 1 and the other starting at 2. Then we shall prove the following theorem.

Theorem 3. *Let* $\Gamma = \Gamma_{k_1,k_2}$ *where* $k_1, k_2 \geq 2$, $m = \min\{k_1, k_2\}$, *and* $M = \max\{k_1, k_2\}$. *Then*

$$NM_\Gamma(t,x,y) = \left(\frac{1}{U_\Gamma(t,y)}\right)^x \text{ where } U_\Gamma(t,y) = 1 + \sum_{n\geq 1} U_{\Gamma,n}(y)\frac{t^n}{n!},$$

$U_{\Gamma,1}(y) = -y$, *and for* $n \geq 2$,

$$U_{\Gamma,n}(y) = (1-y)U_{\Gamma,n-1}(y) - y\binom{n-2}{k_1-1}\left(U_{\Gamma,n-M}(y) + y\sum_{i=1}^{m-1} U_{\Gamma,n-M-i}(y)\right).$$

When $k_1 = k_2 = 2$, Theorem 3 gives us the following corollary.

Corollary 4. *For* $\Gamma = \{1324, 1423\}$, *then*

$$NM_\Gamma(t,x,y) = \left(\frac{1}{U_\Gamma(t,y)}\right)^x \text{ where } U_\Gamma(t,y) = 1 + \sum_{n\geq 1} U_{\Gamma,n}(y)\frac{t^n}{n!},$$

$U_{\Gamma,1}(y) = -y$, *and for* $n \geq 2$,

$$U_{\Gamma,n}(y) = (1-y)U_{\Gamma,n-1}(y) - y(n-2)\left(U_{\Gamma,n-2}(y) + yU_{\Gamma,n-3}(y)\right).$$

Finally, we shall consider families of the form $\Gamma_{k_1,k_2,s} = \Gamma_{k_1,k_2} \cup \{1\cdots s(s+1)\}$ for some $s \geq \max(k_1,k_2)$. For example, we will show that

$$\mathrm{NM}_{\Gamma_{2,2,s}}(t,x,y) = \frac{1}{1+\sum_{n\geq 1} U_{\Gamma_{2,2,s},n}(y)\frac{t^n}{n!}}$$

where $U_{\Gamma_{2,2,s},1}(y) = -y$, and for $n \geq 2$,

$$U_{\Gamma_{2,2,s},n}(y) = -yU_{\Gamma_{2,2,s},n-1}(y) - \sum_{k=0}^{s-2}\left((n-k-1)yU_{\Gamma_{2,2,s},n-k-2}(y) + (n-k-2)y^2U_{\Gamma_{2,2,s},n-k-3}(y)\right).$$

On the surface, it seems that these recursions are more complicated than the recursions for the $U_{\{1324,1423\},n}(y)$'s, but it turns out that the resulting polynomials are considerably simpler to analyze. For example, we shall give explicit formulas for $U_{\Gamma_{2,2,2},n}(y)$ for all $n \geq 1$. That is, we will show that

$$\begin{aligned} U_{\Gamma_{2,2,2},2n}(y) &= \sum_{i=0}^{n}(2n-1)\downarrow\downarrow_{n-i}(-y)^{n+i} \text{ and} \\ U_{\Gamma_{2,2,2},2n+1}(y) &= \sum_{i=0}^{n}(2n)\downarrow\downarrow_{n-i}(-y)^{n+1+i} \end{aligned}$$

where for any x, $(x)\downarrow\downarrow_0 = 1$ and $(x)\downarrow\downarrow_k = x(x-2)(x-4)\cdots(x-2k-2)$ for $k \geq 1$.

The outline of this paper is as follows. In Section 2, we will show how to extend the reciprocity method of Jones and Remmel [13,14] to give combinatorial interpretations to the polynomials $U_{\Gamma,n}(y)$ in the case where all the permutations in Γ start with 1 and have at most one descent. In Section 3, we will prove Theorem 3 and show how to modify it when we add the identity permutation in $\mathfrak{S}_{k+1}$ to the corresponding families in the case where $k_1 = k_2$. In Section 4, we will prove Theorems 1 and 2 and give bijections that will prove our closed expressions for the $U_{\{1324,123\},n}(y)$'s. Finally, in Section 5, we will state some open problems and areas for further research.

2 Symmetric Functions

In this section, we give the necessary background on symmetric functions that will be used in our proofs.

A partition of n is a sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_s)$ such that $0 < \lambda_1 \leq \cdots \leq \lambda_s$ and $n = \lambda_1 + \cdots + \lambda_s$. We shall write $\lambda \vdash n$ to denote that λ is partition of n and we let $\ell(\lambda)$ denote the number of parts of λ. When a partition of n involves repeated parts, we shall often use exponents in the partition notation to indicate these repeated parts. For example, we will write $(1^2, 4^5)$ for the partition $(1, 1, 4, 4, 4, 4, 4)$.

Let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$. The n^{th} elementary symmetric function $e_n = e_n(x_1, x_2, \ldots)$ and n^{th} homogeneous symmetric function $h_n = h_n(x_1, x_2, \ldots)$ are defined by the generating functions given in (2). For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$. It is well known that $e_0, e_1, \ldots$ is an algebraically independent set of generators for Λ, and hence, a ring homomorphism θ on Λ can be defined by simply specifying $\theta(e_n)$ for all n.

If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, then a λ-brick tabloid of shape (n) is a filling of a rectangle consisting of n cells with bricks of sizes $\lambda_1, \ldots, \lambda_k$ in such a way that no two bricks overlap. For example, Figure 1 shows the six $(1^2, 2^2)$-brick tabloids of shape (6).

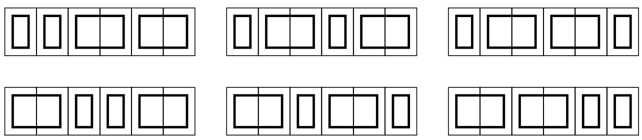

Figure 1: The six $(1^2, 2^2)$-brick tabloids of shape (6).

Let $\mathcal{B}_{\lambda,n}$ denote the set of λ-brick tabloids of shape (n) and let $B_{\lambda,n}$ be the number of λ-brick tabloids of shape (n). If $B \in \mathcal{B}_{\lambda,n}$, we will write $B = (b_1, \ldots, b_{\ell(\lambda)})$ if the lengths of the bricks in B, reading from left to right, are $b_1, \ldots, b_{\ell(\lambda)}$. For example, the brick tabloid in the top right position in Figure 1 is denoted as $(1, 2, 2, 1)$. Eğecioğlu and the second author [6] proved that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n}\, e_\lambda. \tag{3}$$

This interpretation of h_n in terms of e_n will aid us in describing the coefficients of $\Theta_\Gamma(H(t)) = U_\Gamma(t, y)$ described in the next section, which will in turn allow us to compute the coefficients $\mathrm{NM}_{\Gamma,n}(x, y)$.

3 Extending the reciprocity method

In this section, we will show that one can easily extend the reciprocity method of [12–14] to find a combinatorial interpretation for $U_{\Gamma,n}(y)$ in the case where Γ is a set of permutations which all start with 1 and have at most one descent. We can assume that Γ contains at most one permutation σ which is an identity permutation. That is, if $12\cdots s$ and $12\cdots t$ are in Γ for some $s < t$, then if we consecutively avoid $12\cdots s$, we automatically consecutively avoid $12\cdots t$. Thus $\mathcal{NM}_n(\Gamma) = \mathcal{NM}_n(\Gamma - \{12\cdots t\})$ for all n.

We want give a combinatorial interpretation to

$$U_\Gamma(t,y) = \frac{1}{\mathrm{NM}_\Gamma(t,1,y)} = \frac{1}{1+\sum_{n\geq 1}\frac{t^n}{n!}\mathrm{NM}_{\Gamma,n}(1,y)},$$

where

$$\mathrm{NM}_{\Gamma,n}(1,y) = \sum_{\sigma\in\mathcal{NM}_n(\Gamma)} y^{1+\mathrm{des}(\sigma)}.$$

We define a homomorphism Θ_Γ on the ring of symmetric functions Λ by setting $\Theta_\Gamma(e_0) = 1$ and, for $n \geq 1$,

$$\Theta_\Gamma(e_n) = \frac{(-1)^n}{n!}\mathrm{NM}_{\Gamma,n}(1,y).$$

It follows that

$$\begin{aligned}\Theta_\Gamma(H(t)) = \sum_{n\geq 0}\Theta_\Gamma(h_n)t^n &= \frac{1}{\Theta_\tau(E(-t))} = \frac{1}{1+\sum_{n\geq 1}(-t)^n\Theta_\Gamma(e_n)}\\ &= \frac{1}{1+\sum_{n\geq 1}\frac{t^n}{n!}\mathrm{NM}_{\Gamma,n}(1,y)} = U_\Gamma(t,y).\end{aligned}$$

By (3), we have

$$\begin{aligned} n!\Theta_\Gamma(h_n) &= n!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\ \Theta_\Gamma(e_\lambda)\\ &= n!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}\sum_{(b_1,\ldots,b_{\ell(\lambda)})\in\mathcal{B}_{\lambda,n}}\prod_{i=1}^{\ell(\lambda)}\frac{(-1)^{b_i}}{b_i!}\mathrm{NM}_{\Gamma,b_i}(1,y)\\ &= \sum_{\lambda\vdash n}(-1)^{\ell(\lambda)}\sum_{(b_1,\ldots,b_{\ell(\lambda)})\in\mathcal{B}_{\lambda,n}}\binom{n}{b_1,\ldots,b_{\ell(\lambda)}}\prod_{i=1}^{\ell(\lambda)}\mathrm{NM}_{\Gamma,b_i}(1,y). \end{aligned} \tag{4}$$

Next, we want to give a combinatorial interpretation to the right hand side of (4). We select a brick tabloid $B = (b_1, b_2, \ldots, b_{\ell(\lambda)})$ of shape (n) filled with bricks whose sizes induce the partition λ. We interpret the multinomial coefficient $\binom{n}{b_1,\ldots,b_{\ell(\lambda)}}$ as the number of ways to choose an ordered set partition $\mathcal{S} = (S_1, S_2, \ldots, S_{\ell(\lambda)})$ of $\{1, 2, \ldots, n\}$ such that $|S_i| = b_i$ for $i = 1, \ldots, \ell(\lambda)$. For each brick b_i, we then fill the cells of b_i with numbers from S_i such that the entries in the brick reduce to a permutation $\sigma^{(i)} = \sigma_1\cdots\sigma_{b_i}$ in $\mathcal{NM}_{b_i}(\Gamma)$. We label each descent of σ that occurs within each brick as well as the last cell of each brick by y. This accounts for the factor $y^{\mathrm{des}(\sigma^{(i)})+1}$ within each brick. Finally, we use the factor $(-1)^{\ell(\lambda)}$ to change the label of the last cell of each brick from y to $-y$. We will denote the filled labeled brick tabloid constructed in this way as $\langle B, \mathcal{S}, (\sigma^{(1)}, \ldots, \sigma^{(\ell(\lambda))})\rangle$.

For example, when $n = 17, \Gamma = \{1324, 1423, 12345\}$, and $B = (9, 3, 5, 2)$, consider the ordered set partition $\mathcal{S} = (S_1, S_2, S_3, S_4)$ of $\{1, 2, \ldots, 17\}$, where

$$S_1 = \{2, 5, 6, 9, 11, 15, 16, 17, 19\}, S_2 = \{7, 8, 14\}, S_3 = \{1, 3, 10, 13, 18\}, S_4 = \{4, 12\},$$

and the permutations $\sigma^{(1)} = 1\ 2\ 4\ 6\ 5\ 3\ 7\ 9\ 8 \in \mathcal{NM}_9(\Gamma), \sigma^{(2)} = 1\ 3\ 2 \in \mathcal{NM}_7(\Gamma), \sigma^{(3)} = 5\ 1\ 2\ 4\ 3 \in \mathcal{NM}_5(\Gamma)$, and $\sigma^{(4)} = 2\ 1 \in \mathcal{NM}_2(\Gamma)$. The construction of $\langle B, \mathcal{S}, (\sigma^{(1)}, \ldots, \sigma^{(4)})\rangle$ is then pictured in Figure 2.

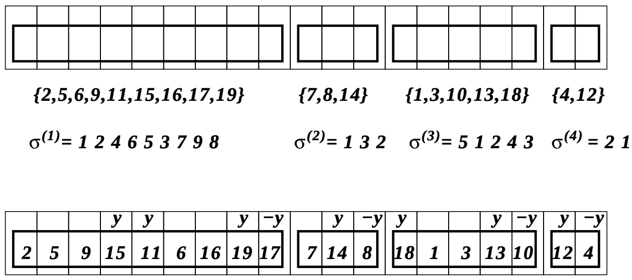

Figure 2: The construction of a filled-labeled-brick tabloid.

It is easy to see that we can recover the triple $\langle B, (S_1, \ldots, S_{\ell(\lambda)}), (\sigma^{(1)}, \ldots, \sigma^{(\ell(\lambda))})\rangle$ from B and the permutation σ which is obtained by reading the entries in the cells from right to left. We let $\mathcal{O}_{\Gamma,n}$ denote the set of all filled labeled brick tabloids created this way. That is, $\mathcal{O}_{\Gamma,n}$ consists of all pairs $O = (B, \sigma)$ where

1. $B = (b_1, b_2, \ldots, b_{\ell(\lambda)})$ is a brick tabloid of shape n,
2. $\sigma = \sigma_1 \cdots \sigma_n$ is a permutation in $\mathfrak{S}_n$ such that there is no Γ-match of σ which lies entirely in a single brick of B, and
3. if there is a cell c such that a brick b_i contains both cells c and $c + 1$ and $\sigma_c > \sigma_{c+1}$, then cell c is labeled with a y and the last cell of any brick is labeled with $-y$.

We define the sign of each O to be $sgn(O) = (-1)^{\ell(\lambda)}$. The weight $W(O)$ of O is defined to be the product of all the labels y used in the brick. Thus, the weight of the filled labeled brick tabloid from Figure 2 above is $W(O) = y^{11}$. It follows that

$$n!\Theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}} sgn(O)W(O). \tag{5}$$

Following [13], we next define a sign-reversing, weight-preserving involution $I : \mathcal{O}_{\Gamma,n} \to \mathcal{O}_{\Gamma,n}$. Given a filled labeled brick tabloid $(B, \sigma) \in \mathcal{O}_{\Gamma,n}$ where $B = (b_1, \ldots, b_k)$, we read the cells of (B, σ) from left to right, looking for the first cell c for which either

(i) cell c is labeled with a y, or

(ii) cell c is at the end of brick b_i where $\sigma_c > \sigma_{c+1}$ and there is no Γ-match of σ that lies entirely in the cells of the bricks b_i and b_{i+1}.

In case (i), we define $I_\Gamma(B,\sigma)$ to be the filled labeled brick tabloid obtained from (B,σ) by breaking the brick b_j that contains cell c into two bricks b'_j and b''_j where b'_j contains the cells of b_j up to and including the cell c while b''_j contains the remaining cells of b_j. In addition, we change the labeling of cell c from y to $-y$. In case (ii), $I_\Gamma(B,\sigma)$ is obtained by combining the two bricks b_i and b_{i+1} into a single brick b and changing the label of cell c from $-y$ to y. If neither case occurs, then we let $I_\Gamma(B,\sigma) = (B,\sigma)$.

For instance, the image of the filled labeled brick tabloid from the Figure 2 under this involution is shown below in Figure 3.

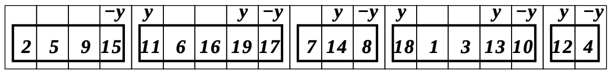

Figure 3: $I_\Gamma(O)$ for O in Figure 2.

We claim that as long as each permutation in Γ has at most one descent, then I_Γ is an involution. Let (B,σ) be an element of $\mathcal{O}_{\gamma,n}$ which is not a fixed point of I. Suppose that $I(B,\sigma)$ is defined using case (i) where we split a brick b_j at cell c which is labeled with a y. In that case, we let a be the number in cell c and a' be the number in cell $c+1$ which must also be in brick b_j. Since cell c is labeled with y, it must be the case that $a > a'$. Moreover, there can be no cell labeled y that occurs before cell c since otherwise we would not use cell c to define $I(B,\sigma)$. In this case, we must ensure that when we split b_j into b'_j and b''_j, we cannot combine the brick b_{j-1} with b'_j because the number in that last cell of b_{j-1} is greater than the number in the first cell of b'_j and there is no Γ-match in the cells of b_{j-1} and b'_j since in such a situation, $I_\Gamma(I_\Gamma(B,\sigma)) \neq (B,\sigma)$. However, since we always take an action on the leftmost cell possible when defining $I_\Gamma(B,\sigma)$, we know that we cannot combine b_{j-1} and b_j so that there must be a Γ-match in the cells of b_{j-1} and b_j. Moreover, if we could now combine bricks b_{j-1} and b'_j, then that Γ-match must have involved the number a' and the number in cell d which is the last cell in brick b_{j-1}. But that is impossible because then there would be two descents among the numbers between cell d and cell $c+1$ which would violate our assumption that the elements of Γ have at most one descent. Thus whenever we apply case (i) to define $I_\Gamma(B,\sigma)$, the first action that we can take is to combine bricks b'_j and b''_j so that $I^2_\Gamma(B,\sigma) = (B,\sigma)$.

If we are in case (ii), then again we can assume that there are no cells labeled y that occur before cell c. When we combine brick b_i and b_{i+1}, then we will label cell c with a y. It is clear that combining the cells of b_i and b_{i+1} cannot help us combine the resulting brick b with b_{j-1} since, if there were a Γ-match that prevented us from combining bricks b_{j-1} and b_j, then that same Γ-match will prevent us from combining b_{j-1} and b. Thus, the first place where we can apply the involution will again be cell c which is now labeled with a y so that $I^2_\Gamma(B,\sigma) = (B,\sigma)$.

It is clear that if $I_\Gamma(B,\sigma) \neq (B,\sigma)$, then

$$sgn(B,\sigma)W(B,\sigma) = -sgn(I_\Gamma(B,\sigma))W(I_\Gamma(B,\sigma)).$$

Thus it follows from (5) that

$$n!\Theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}} \operatorname{sgn}(O)W(O) = \sum_{O \in \mathcal{O}_{\Gamma,n}, I_\Gamma(O)=O} \operatorname{sgn}(O)W(O).$$

Hence if all permutations in Γ have at most one descent, then

$$U_{\Gamma,n}(y) = \sum_{O \in \mathcal{O}_{\Gamma,n}, I_\Gamma(O)=O} \operatorname{sgn}(O)W(O). \tag{6}$$

Thus to compute $U_{\Gamma,n}(y)$, we must analyze the fixed points of I_Γ.

If (B,σ) where $B = (b_1,\ldots,b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$ is a fixed point of the involution I_Γ, then (B,σ) cannot have any cell labeled y which means that the elements of σ that lie within any brick b_j of B must be increasing. If it is the case that an identity permutation $12\cdots(k+1)$ is in Γ, then no brick of B can have length greater than k. Next, consider any two consecutive bricks b_i and b_{i+1} in B. Let c be the last cell of b_i and $c+1$ be the first cell of b_{i+1}. Then either $\sigma_c < \sigma_{c+1}$ in which case we say there is an increase between bricks b_i and b_{i+1}, or $\sigma_c > \sigma_{c+1}$ in which case we say there is a decrease between bricks b_i and b_{i+1}. In the latter case, there must be a Γ-match of σ that lies in the cells of b_i and b_{i+1} which must necessarily involve σ_c and σ_{c+1}. Finally, we claim that since all the permutations in Γ start with 1, the minimal elements within the bricks of B must increase from left to right. That is, consider two consecutive bricks b_i and b_{i+1} and let c_i and c_{i+1} be the first cells of b_i and b_{i+1}, respectively. Suppose that $\sigma_{c_i} > \sigma_{c_{i+1}}$. Let d_i be the last cell of b_i. Then clearly $\sigma_{c_{i+1}} < \sigma_{c_i} \leq \sigma_{d_i}$ so that there is a decrease between brick b_i and brick b_{i+1} and hence there must be a Γ-match of σ that lies in the cells of b_i and b_{i+1} that involves the elements of σ_{d_i} and $\sigma_{c_{i+1}}$. But this is impossible since our assumptions ensure that $\sigma_{c_{i+1}}$ is the smallest element that lies in the bricks b_i and b_{i+1} so that it can only play the role of 1 in any Γ-match. But since every element of Γ starts with 1, then any Γ-match that lies in b_i and b_{i+1} that involves $\sigma_{c_{i+1}}$ must lie entirely in brick b_{i+1} which contradicts the fact that (B,σ) was a fixed point of I_Γ.

Thus, we have the following lemma describing the fixed points of the involution I_Γ.

Lemma 5. *Let Γ be a set of permutations which all start with 1 and have at most one descent. Let $\mathbb{Q}(y)$ be the set of rational functions in the variable y over the rationals $\mathbb{Q}$ and let $\Theta_\Gamma : \Lambda \to \mathbb{Q}(y)$ be the ring homomorphism defined by setting $\Theta_\Gamma(e_0) = 1$, and $\Theta_\Gamma(e_n) = \frac{(-1)^n}{n!} NM_{\Gamma,n}(1,y)$ for $n \geq 1$. Then*

$$n!\Theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}, I_\Gamma(O)=O} \operatorname{sgn}(O)W(O)$$

where $\mathcal{O}_{\Gamma,n}$ is the set of objects and I_Γ is the involution defined above. Moreover, $O = (B,\sigma) \in \mathcal{O}_{\Gamma,n}$ where $B = (b_1,\ldots,b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$ is a fixed point of I_Γ if and only if O satisfies the following four properties:

1. *there are no cells labeled with y in O, i.e., the elements in each brick of O are increasing,*

2. *the first elements in each brick of O form an increasing sequence, reading from left to right,*

3. *if b_i and b_{i+1} are two consecutive bricks in B, then either (a) there is increase between b_i and b_{i+1}, i.e., $\sigma_{\sum_{j=1}^{i} b_j} < \sigma_{1+\sum_{j=1}^{i} b_j}$, or (b) there is a decrease between b_i and b_{i+1}, i.e., $\sigma_{\sum_{j=1}^{i} b_j} > \sigma_{1+\sum_{j=1}^{i} b_j}$, and there is a Γ-match contained in the elements of the cells of b_i and b_{i+1} which must necessarily involve $\sigma_{\sum_{j=1}^{i} b_j}$ and $\sigma_{1+\sum_{j=1}^{i} b_j}$, and*

4. *if Γ contains an identity permutation $12\cdots(k+1)$, then $b_i \leq k$ for all i.*

Note that since $U_{\Gamma,n}(y) = n!\Theta_\Gamma(h_n)$, Lemma 5 gives us a combinatorial interpretation of $U_{\Gamma,n}(y)$. Since the weight of of any fixed point (B,σ) of I_Γ is $-y$ raised to the number of bricks in B, it follows that $U_{\Gamma,n}(-y)$ is always a polynomial with non-negative integer coefficients. We will exploit this combinatorial interpretation to prove the main results of this paper.

4 Proof of Theorem 3

Let $k_1, k_2 \geq 2$ and $p = k_1 + k_2$. We consider the family of permutations $\Gamma = \Gamma_{k_1,k_2}$ in $\mathfrak{S}_p$ where

$$\Gamma_{k_1,k_2} = \{\sigma \in \mathfrak{S}_p : \sigma_1 = 1, \sigma_{k_1+1} = 2, \sigma_1 < \sigma_2 < \cdots < \sigma_{k_1} \ \&\ \sigma_{k_1+1} < \sigma_{k_1+2} < \cdots < \sigma_p\}.$$

We start this section by giving a proof of Theorem 3. At the end of this section, we shall consider how to compute $U_{\Gamma_{k_1,k_1,s}}(y,t)$ where

$$\Gamma_{k_1,k_1,s} = \Gamma_{k_1,k_1} \cup \{12\cdots s(s+1)\}.$$

By (6), we must show that the coefficients

$$U_{\Gamma,n}(y) = \sum_{O \in \mathcal{O}_{\Gamma,n}, I_\Gamma(O)=O} \operatorname{sgn}(O)W(O)$$

have the following properties:

1. $U_{\Gamma,1}(y) = -y$, and

2. for $n > 1$,

$$U_{\Gamma,n}(y) = (1-y)U_{\Gamma,n-1}(y) - y\binom{n-2}{k_1-1}\left(U_{\Gamma,n-M}(y) + y\sum_{i=1}^{m-1} U_{\Gamma,n-M-i}(y)\right),$$

where $m = \min\{k_1, k_2\}$ and $M = \max\{k_1, k_2\}$.

We will divide the proof into two cases, one where $k_1 \geq k_2$ and the other where $k_1 < k_2$.

Case I. $k_1 \geq k_2$.

Let (B, σ) be a fixed point of I_Γ where $B = (b_1, \ldots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$. We know that 1 is in the first cell of (B, σ). We claim that 2 must be in cell 2 or cell k_1+1 of (B, σ). To see this, suppose that 2 is in cell c where $c \neq 2$ and $c \neq k_1 + 1$. Since there is no descent within any brick, 2 must be the first cell of its brick. Moreover, since the minimal elements of the bricks form an increasing sequence, reading from left to right, 2 must be in the first cell of the second brick b_2. Thus, 1 is in the first cell of the first brick b_1 and 2 is in the first cell of the second brick b_2. Since $c > 2$, there is a decrease between bricks b_1 and b_2 and, hence, there must be a Γ-match of σ contained cells of b_1 and b_2 which involves 2 and the last cell of b_1. Since all the elements of Γ start with 1, this Γ-match must also involve 1 since only 1 can play the role of 1 in a Γ-match that involves 2 and the last cell of b_1. But in all such Γ-matches, 2 must be in cell $k_1 + 1$. Since $c \neq k_1 + 1$, this means that there can be no Γ-match contained in the cells of b_1 and b_2 which contradicts the fact that (B, σ) is a fixed point of I_Γ.

Thus, we have two subcases.

Subcase 1. 2 is in cell 2 of (B, σ).

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of (B, σ) or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of (B, σ). In either case, we know that 1 is not part of a Γ-match in (B, σ). So if we remove cell 1 from (B, σ) and subtract 1 from the elements in the remaining cells, we will obtain a fixed point O' of I_Γ in $\mathcal{O}_{\Gamma,n-1}$.

Moreover, we can create a fixed point $O = (B, \sigma) \in \mathcal{O}_n$ satisfying conditions (1), (2), (3) and (4) of Lemma 5 where $\sigma_2 = 2$ by starting with a fixed point $(B', \sigma') \in \mathcal{O}_{\Gamma,n-1}$ of I_Γ, where $B' = (b'_1, \ldots, b'_r)$ and $\sigma' = \sigma'_1 \cdots \sigma'_{n-1}$, and then letting $\sigma = 1(\sigma'_1 + 1) \cdots (\sigma'_{n-1} + 1)$, and setting $B = (1, b'_1, \ldots, b'_r)$ or setting $B = (1 + b'_1, \ldots, b'_r)$.

It follows that fixed points in Case 1 will contribute $(1 - y)U_{\Gamma,n-1}(y)$ to $U_{\Gamma,n}(y)$.

Subcase 2. 2 is in cell $k_1 + 1$ of (B, σ).

Since there is no decrease within the bricks of (B, σ) and the first numbers of the bricks are increasing, reading from left to right, it must be the case that 2 is in the first cell of b_2. Thus b_1 has exactly k_1 cells. In addition, b_2 has at least k_2 cells since otherwise, there could be no Γ-match contained in the cells of b_1 and b_2 and we could combine the bricks b_1 and b_2, which would mean that (B, σ) is not a fixed point of I_Γ. By our argument above, it must be the case that the Γ-match of σ contained in the cells of b_1 and b_2 must start in the first cell. We first choose $k_1 - 1$ numbers to fill in the remaining cells of b_1. There are $\binom{n-2}{k_1-1}$ ways to do this. For each such choice, we let O' be the result by removing the first k_1 cells from (B, σ) and replacing the i^{th} largest remaining number by i for $i = 1, \ldots, n - k_1$, then O' will be a fixed point in $\mathcal{O}_{\Gamma,n-k_1}$ whose first brick is of size greater than or equal to k_2.

On the other hand, suppose that we start with $O' \in \mathcal{O}_{\Gamma,n-k_1}$ which is a fixed point of I_Γ and whose first brick is of size greater than or equal to k_2. Then we can take any $k_1 - 1$ numbers $1 < a_1 < a_2 < \cdots < a_{k_1-1} \leq n$ and add a new brick at the start which contains $1, a_1, \ldots a_{k_1-1}$ followed by O'' which is the result of replacing the numbers in O' by the numbers in $\{1,\ldots,n\} - \{1, a_1, \ldots a_{k_1-1}\}$ maintaining the same relative order, then we will create a fixed point O of I_Γ of size n whose first brick is of size k_1 and whose second brick starts with 2.

Thus we need to count the number of fixed points in $\mathcal{O}_{\Gamma,n-k_1}$whose first brick has size at least k_2. Suppose that $V = (D, \tau)$ is a fixed point of $\mathcal{O}_{\Gamma,n-k_1}$ where $D = (d_1, \ldots, d_k)$ and $\tau = \tau_1 \cdots \tau_{n-k_1}$. Now if $d_1 = j < k_2$, then there cannot be a decrease between bricks d_1 and d_2 because otherwise there would have been a Γ-match starting at cell 1 contained in the bricks d_1 and d_2 which is impossible since all permutations in Γ have their only descent at position $k_1 > j$. This means that the first brick d_1 must be filled with $1, \ldots j$. That is, since the minimal elements of the bricks are increasing reading from left to right, we must have that the first element of d_2, namely τ_{j+1}, is less than all the elements to its right and we have shown that all the elements in the first brick are less than τ_{j+1}. It follows that $\tau_1 \cdots \tau_{j+1} = 12 \cdots j(j+1)$. Therefore, if we let V' be the result of removing the entire first brick of V and subtracting j from the remaining numbers, then V' is a fixed point in $\mathcal{O}_{\Gamma,n-k_1-j}$.

It follows that

$$U_{\Gamma,n-k_1}(y) - \sum_{j=1}^{k_2-1} (-y) U_{\Gamma,n-k_1-j}(y)$$

equals the sum over all fixed points of $I_{\Gamma,n-k_1}$ whose first brick has size at least k_2. Hence the contribution of fixed points in Case 2 to $U_{\Gamma,n}(y)$ is

$$(-y)\binom{n-2}{k_1-1}\left(U_{\Gamma,n-k_1}(y) + \sum_{j=1}^{k_2-1} yU_{\Gamma,n-k_1-j}(y)\right).$$

Combining the two cases, we see that for $n > 1$,

$$U_{\Gamma,n}(y) = (1-y)U_{\Gamma,n-1}(y) - y\binom{n-2}{k_1-1}\left(U_{\Gamma,n-k_1}(y) + y\sum_{j=1}^{k_2-1} U_{\Gamma,n-k_1-i}(y)\right). \quad (7)$$

Case II. $k_1 < k_2$.

Let $O = (B, \sigma)$ be a fixed point of I_Γ where $B = (b_1, \ldots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$. We know that 1 is in the first cell of O. By the same argument as in Case I, we know that 2 must be in cell 2 or cell $k_1 + 1$ of O. We now consider two cases depending on the position of 2 in O.

Subcase A. 2 is in cell 2 of (B, σ).

By the same argument that we used in Subcase 1 of Case I, we can conclude that the fixed points of I_Γ in Subcase A will contribute $(1-y)U_{\Gamma,n-1}(y)$ to $U_{\Gamma,n}(y)$.

Subcase B. 2 is in cell k_1+1 of (B,σ).

Since the minimal elements of the bricks are increasing, reading from left to right, it must be the case that 2 is in the first cell of b_2. Thus, b_1 has exactly k_1 cells, b_2 has at least k_2 cells, and there is a Γ_{k_1,k_2}-match in the cells of b_1 and b_2 which must start at cell 1.

We first choose k_1-1 numbers to fill in the remaining cells of b_1. There are $\binom{n-2}{k_1-1}$ ways to do this. For each of such choice, let $d_1 < \cdots < d_{k_2-k_1-1}$ be the smallest k_2-k_1-1 numbers in $\{1,2,\ldots,n\}-\{\sigma_1,\ldots,\sigma_{k_1+1}\}$. We claim that it must be the case that $\sigma_{k_1+1+i}=d_i$ for $i=1,\ldots,k_2-k_1-1$. If not, let j be the least i such that $\sigma_{k_1+1+i}\neq d_i$. Then d_i cannot be in brick b_2 so that it must be the first element in brick b_3. But then there will be a decrease between bricks b_2 and b_3 which means that there must be a Γ_{k_1,k_2}-match contained in the cells of b_2 and b_3. Note that there is only one descent in each permutation of Γ_{k_1,k_2} and this descent must occur at position k_1. It follows that this Γ_{k_1,k_2}-match must start at the $(k_2-k_1)^{th}$ cell of b_2. But this is impossible since our assumption will ensure that $\sigma_{k_1+1+(k_2-k_1-1)}=\sigma_{k_2}>d_i$.

It then follows that if we let O' be the result by removing the first k_2 cells from O and adjusting the remaining numbers in the cells, then O' will be a fixed point in $\mathcal{O}_{\Gamma,n-k_2}$ that starts with at least k_1 cells in the first brick. Then we can argue exactly as we did in Subcase 2 of Case I the contribution of fixed points in Case B to $U_{\Gamma,n}(y)$ is

$$-y\binom{n-2}{k_1-1}\left(U_{\Gamma,n-k_2}(y)+\sum_{j=1}^{k_1-1}yU_{\Gamma,n-k_2-j}(y)\right).$$

It follows that in Case II

$$U_{\Gamma_{k_1,k_2},n}(y)=(1-y)U_{\Gamma_{k_1,k_2},n}(y)-y\binom{n-2}{k_1-1}\left(U_{\Gamma,n-k_2}(y)+\sum_{j=1}^{k_1-1}yU_{\Gamma,n-k_2-j}(y)\right) \quad (8)$$

for $n>1$.

Comparing equations (7) and (8), it is easy to see that if $m=\min(k_1,k_2)$ and $M=\max(k_1,k_2)$, then

$$U_{\Gamma_{k_1,k_2},n}(y)=(1-y)U_{\Gamma_{k_1,k_2},n-1}(y)-y\binom{n-2}{k_1-1}\left(U_{\Gamma,n-M}(y)+y\sum_{i=1}^{m-1}U_{\Gamma,n-M-i}(y)\right)$$

for all $n>1$ which proves Theorem 3.

For example, consider the special case where $k_1=k_2=2$. Then by Corollary 4,

$$U_{\Gamma_{2,2},n}(y)=(1-y)U_{\Gamma_{2,2},n-1}(y)-y(n-2)\left(U_{\Gamma_{2,2},n-2}(y)+yU_{\Gamma_{2,2},n-3}(y)\right).$$

In Table 1, we computed $U_{\Gamma_{2,2},n}(y)$ for $n\leq 14$.

We observe that the polynomials $U_{\Gamma_{2,2},n}(-y)$ in Table 1 are all log-concave. Here, a polynomial $P(y)=a_0+a_1y+\cdots+a_ny^n$ is called *log-concave* if $a_{i-1}a_{i+1}<a_i^2$, for all $i=2,\ldots,n-1$, and it is called *unimodal* if there exists an index k such that

n	$U_{\Gamma_{2,2},n}(-y)$
1	y
2	$y+y^2$
3	$y+2y^2+y^3$
4	$y+5y^2+3y^3+y^4$
5	$y+9y^2+11y^3+4y^4+y^5$
6	$y+14y^2+36y^3+19y^4+5y^5+y^6$
7	$y+20y^2+90y^3+85y^4+29y^5+6y^6+y^7$
8	$y+27y^2+188y^3+337y^4+162y^5+41y^6+7y^7+y^8$
9	$y+35y^2+348y^3+1057y^4+842y^5+273y^6+55y^7+8y^8+y^9$
10	$y+44y^2+591y^3+2749y^4+3875y^5+1731y^6+424y^7+71y^8+9y^9+y^{10}$
11	$y+54y^2+941y^3+6229y^4+14445y^5+10151y^6+3154y^7+621y^8$ $+89y^9+10y^{10}+y^{11}$
12	$y+65y^2+1425y^3+12730y^4+44684y^5+52776y^6+22195y^7+5285y^8$ $+870y^9+109y^{10}+11y^{11}+y^{12}$
13	$y+77y^2+2073y^3+24022y^4+119432y^5+226116y^6+144007y^7+43133y^8$ $+8322y^9+1177y^{10}+131y^{11}+12y^{12}+y^{13}$
14	$y+90y^2+2918y^3+42547y^4+284922y^5+807008y^6+830095y^7+331668y^8$ $+77027y^9+12487y^{10}+1548y^{11}+155y^{12}+13y^{13}+y^{14}$

Table 1: The polynomials $U_{\Gamma_{2,2},n}(-y)$ for $\Gamma_{2,2}=\{1324,1423\}$

$a_i \le a_{i+1}$ for $1 \le i \le k-1$ and $a_i \ge a_{i+1}$ for $k \le i \le n-1$. We conjecture that the polynomials $U_{\Gamma_{2,2},n}(-y)$ are log-concave and, hence, unimodal for all n. We checked this holds for $n \le 21$.

One might hope to prove the unimodality of the polynomials $U_{\Gamma_{2,2},n}(-y)$ by using the recursion

$$U_{\Gamma_{2,2},n}(-y) = (1+y)U_{\Gamma_{2,2},n-1}(-y)+(n-2)yU_{\Gamma_{2,2},n2}(-y)+(n-2)y^2U_{\Gamma_{2,2},n-3}(-y) \quad (9)$$

and showing that for large enough n, the polynomials on the right hand side of (9) are all unimodal polynomials whose maximum coefficients occur at the same power of y. There are two problems with this idea. First, assuming that $U_{\Gamma_{2,2},n}(-y)$ is a unimodal polynomial whose maximum coefficient occurs that y^j, then we know that $(1+y)U_{\Gamma_{2,2},n}(-y)$ is a unimodal polynomial. However, it could be that the maximum coefficient of $(1+y)U_{\Gamma_{2,2},n}(-y)$ occurs at y^j or at y^{j+1}. That is, if $P(y)$ is a unimodal polynomial whose maximum coefficient occurs at y^k, then $(1+y)P(y)$ could have its maximum coefficient occur at either y^k or y^{k+1}. For example,

$$(1+y)(1+5y+2y^2) = 1+6y+7y^2+2y^3$$

while

$$(1+y)(2+5y+y^2) = 2+7y+6y^2+y^3.$$

Thus where the maximum coefficient of $(1+y)U_{\Gamma_{2,2},n}(-y)$ occurs depends on the relative values of the coefficients on either side of the maximum coefficient of $U_{\Gamma_{2,2},n}(-y)$.

For $n \leq 20$, the maximum coefficient of $(1+y)U_{\Gamma_{2,2},n}(-y)$ occurs at the same power of y where the maximum coefficient of $U_{\Gamma_{2,2},n}(-y)$ occurs, but it is not obvious that this holds for all n.

Second, it is not clear where to conjecture the maximum coefficients in the polynomials occur. That is, one might think from the table that for $n \geq 6$, the maximum coefficient in $U_{\Gamma_{2,2},n}(-y)$ occurs at $y^{\lfloor n/2 \rfloor +1}$, but this does not hold up. For example, the maximum coefficient $U_{\Gamma_{2,2},18}(-y)$ occurs at y^8 and the maximum coefficient $U_{\Gamma_{2,2},19}(-y)$ occurs at y^9. Moreover, the maximum coefficient $U_{\Gamma_{2,2},26}(-y)$ occurs at y^{12} and the maximum coefficient $U_{\Gamma_{2,2},27}(-y)$ occurs at y^{12}. Thus it is not clear how to use the recursion (9) to even prove the unimodality of the polynomials $U_{\Gamma_{2,2},n}(-y)$ much less prove that such polynomials are log concave.

When k_1 is larger than k_2, the polynomials $U_{\Gamma_{k_1,k_2},n}(-y)$ are not always unimodal. For example, consider the case where $k_1 = 6$ and $k_2 = 4$. Mathematica once again allows us to compute $U_{\Gamma_{6,4},n}(-y)$ for $n = 10$ and 11. It is quite easy to see from Table 2 that neither polynomial is unimodal.

n	$U_{\Gamma_{6,4},n}(-y)$
10	$y+65y^2+36y^3+84y^4+126y^5+126y^6+84y^7+36y^8+9y^9+y^{10}$
11	$y+192y^2+227y^3+120y^4+210y^5+252y^6+210y^7+120y^8+45y^9+10y^{10}+y^{11}$

Table 2: The polynomials $U_{\Gamma_{6,4},n}(-y)$

4.1 Adding an identity permutation to Γ_{k_1,k_2}

In this subsection, we want to consider the effect of adding an identity permutation to Γ_{k_1,k_2}. To simplify our analysis, we shall consider only the case where $k_1 = k_2$, but the same type of analysis can be carried out in general. Thus assume that $s \geq k_1 = k_2 \geq 2$ and let $\Gamma_{k_1,k_1,s} = \Gamma_{k_1,k_1} \cup \{12\cdots s(s+1)\}$. Then we know that

$$U_{\Gamma_{k_1,k_1,s},n}(y) = \sum_{O \in \mathcal{O}_{\Gamma_{k_1,k_1,s},n},\ I_{\Gamma_{k_1,k_1,s}}(O)=O} \operatorname{sgn}(O)W(O).$$

We want to classify the fixed points of $I_{\Gamma_{k_1,k_1,s}}$ by the size of the first brick. By Lemma 5, it must be the case that the size of the first brick is less than or equal to s. We let $U^{(r)}_{\Gamma_{k_1,k_1,s},n}(y)$ denote the sum of $\operatorname{sgn}(O)W(O)$ over all fixed points of $I_{\Gamma_{k_1,k_1,s}}$ whose first brick is of size r. Thus,

$$U_{\Gamma_{k_1,k_1,s},n}(y) = \sum_{r=1}^{s} U^{(r)}_{\Gamma_{k_1,k_1,s},n}(y). \tag{10}$$

Now let $O = (B,\sigma)$ be a fixed point of $I_{\Gamma_{k_1,k_1,s}}$ where $B = (b_1,\ldots,b_k)$ and $\sigma = \sigma_1\cdots\sigma_n$. By our arguments above, if $b_1 < k_1$, then the elements in the first brick of (B,σ) are $1,\ldots,b_1$ so that for $1 \leq r < k_1$,

$$U^{(r)}_{\Gamma_{k_1,k_1,s},n}(y) = -yU_{\Gamma_{k_1,k_1,s},n-r}(y). \tag{11}$$

Let

$$U^{(\geq k_1)}_{\Gamma_{k_1,k_1,s},n}(y) = \sum_{r=k_1}^{s} U^{(r)}_{\Gamma_{k_1,k_1,s},n}(y)$$

be the sum of $\mathrm{sgn}(O)W(O)$ over all fixed points of $I_{\Gamma_{k_1,k_1,s}}$ whose first brick has size greater than or equal to k_1. Clearly,

$$\begin{aligned} U_{\Gamma_{k_1,k_1,s},n}(y) &= U^{(\geq k_1)}_{\Gamma_{k_1,k_1,s},n}(y) + \sum_{r=1}^{k_1-1} U^{(r)}_{\Gamma_{k_1,k_1,s},n}(y) \\ &= U^{(\geq k_1)}_{\Gamma_{k_1,k_1,s},n}(y) + \sum_{r=1}^{k_1-1} (-y) U_{\Gamma_{k_1,k_1,s},n-r}(y) \end{aligned}$$

so that

$$U^{(\geq k_1)}_{\Gamma_{k_1,k_1,s},n}(y) = U_{\Gamma_{k_1,k_1,s},n}(y) + \sum_{r=1}^{k_1-1} y U_{\Gamma_{k_1,k_1,s},n-r}(y). \tag{12}$$

Now suppose that $r > k_1$. Then we claim that $\sigma_i = i$ for $i = 1, \ldots, r-k_1+1$. That is, we know that $\sigma_1 = 1$ so that if it is not the case that $\sigma_i = i$ for $i = 1, \ldots, r-k_1+1$, there must be a least $i \leq r-k_1+1$ which is not in the first brick of (B, σ). Since there are no descents of σ within bricks and the minimal elements of the bricks of (B, σ) are increasing, reading from left to right, it must be that i is the first element of brick b_2 and there is a decrease between bricks b_1 and b_2. Thus there is a $\Gamma_{k_1,k_1,s}$-match that lies in the cells of b_1 and b_2 and the only place that such a match can start is at cell $r - k_1 + 1$. But this is impossible since we would have $\sigma_{r-k_1+1} > i$ which is incompatible with having a $\Gamma_{k_1,k_1,s}$-match starting at cell $r-k_1+1$. It follows that we can remove the first $r - k_1$ elements from (B, σ) and reduce the remaining elements by $r - k_1$ to produce a fixed point of $I_{\Gamma_{k_1,k_1,s}}$ of size $n - (r - k_1)$ whose first brick has size k_1. Vice versa, if we start with a fixed point (D, τ) of $I_{\Gamma_{k_1,k_1,s}}$ of size $n - (r - k_1)$ where $D = (d_1, \ldots, d_k)$, $\tau = \tau_1 \cdots \tau_{n-(r-k_1)}$, and $d_1 = k_1$, then if we add $1, \ldots, r - k_1$ to the first brick and raise the remaining numbers by $r - k_1$, we will produce a fixed point of $I_{\Gamma_{k_1,k_1,s}}$ whose first brick is of size r. It follows that for $k_1 < r \leq s$,

$$U^{(r)}_{\Gamma_{k_1,k_1,s},n}(y) = U^{(k_1)}_{\Gamma_{k_1,k_1,s},n-(r-k_1)}(y). \tag{13}$$

Thus

$$U^{(\geq k_1)}_{\Gamma_{k_1,k_1,s},n}(y) = \sum_{p=0}^{s-k_1} U^{(k_1)}_{\Gamma_{k_1,k_1,s},n-p}(y). \tag{14}$$

Finally consider $U^{(k_1)}_{\Gamma_{k_1,k_1,s},n}(y)$. Let (B, σ) be a fixed point of $I_{\Gamma_{k_1,k_1,s}}$ where $B = (b_1, \ldots, b_k)$, $b_1 = k_1$, and $\sigma = \sigma_1 \cdots \sigma_n$. We then have two cases.

Case 1. 2 is in brick b_1.

In this case, we claim that the first brick must contain the elements $1, \ldots, k_1$. That is, in such a situation 1 cannot be involved in a $\Gamma_{k_1,k_1,s}$-match in σ which means that there is not enough room for a $\Gamma_{k_1,k_1,s}$-match that involves any elements from the first brick. Thus as before, we can remove the first brick from (B, σ) and subtract k_1 from the remaining elements of σ to produce a fixed point (D, τ) of $I_{\Gamma_{k_1,k_1,s}}$ of size $n-k_1$. Such fixed points contribute $(-y)U_{\Gamma_{k_1,k_1,s},n-k_1}(y)$ to $U^{(k_1)}_{\Gamma_{k_1,k_1,s},n}(y)$.

Case 2. 2 is in brick b_2.

In this case, we can argue as above that 2 be the first cell of the second brick b_2 and b_2 starts at cell k_1+1. Then we have $\binom{n-2}{k_1-1}$ ways to choose the remaining elements in the first brick and if we remove the first brick and adjust the remaining elements, we will produce a fixed point (D, τ) of $I_{\Gamma_{k_1,k_1,s}}$ of size $n-k_1$ whose first brick is of size greater than or equal to k_1. Such fixed points contribute $(-y)\binom{n-2}{k_1-1}U^{(\geq k_1)}_{\Gamma_{k_1,k_1,s},n-k_1}(y)$ to $U^{(k_1)}_{\Gamma_{k_1,k_1,s},n}(y)$.

It follows that

$$\begin{aligned} U^{(k_1)}_{\Gamma_{k_1,k_1,s},n}(y) &= -yU_{\Gamma_{k_1,k_1,s},n-k_1}(y) - y\binom{n-2}{k_1-1}U^{(\geq k_1)}_{\Gamma_{k_1,k_1,s},n-k_1}(y) \\ &= -yU_{\Gamma_{k_1,k_1,s},n-k_1}(y) - \\ &\quad y\binom{n-2}{k_1-1}\left(U_{\Gamma_{k_1,k_1,s},n-k_1}(y) + y\sum_{r=1}^{k_1-1}U_{\Gamma_{k_1,k_1,s},n-k_1-r}(y)\right). \end{aligned} \tag{15}$$

Putting equations (10), (11), (12), (13), (14), and (15) together, we see that

$$\begin{aligned} &U_{\Gamma_{k_1,k_1,s},n}(y) \\ &= -y\sum_{r=1}^{k_1-1}U_{\Gamma_{k_1,k_1,s},n-r}(y) + \sum_{p=0}^{s-k_1}U^{(k_1)}_{\Gamma_{k_1,k_1,s},n-p}(y) \\ &= -y\sum_{r=1}^{k_1-1}U_{\Gamma_{k_1,k_1,s},n-r}(y) - y\sum_{p=0}^{s-k_1}U_{\Gamma_{k_1,k_1,s},n-p-k_1}(y) \\ &\qquad + \binom{n-p-2}{k_1-1}\left(U_{\Gamma_{k_1,k_1,s},n-p-k_1}(y) + y\sum_{a=1}^{k_1-1}U_{\Gamma_{k_1,k_1,s},n-p-k_1-a}(y)\right) \\ &= -y\sum_{r=1}^{k_1-1}U_{\Gamma_{k_1,k_1,s},n-r}(y) - y\left(\sum_{p=0}^{s-k_1}\left(1+\binom{n-p-2}{k_1-1}\right)U_{\Gamma_{k_1,k_1,s},n-p-k_1}(y)\right. \\ &\qquad \left. + y\binom{n-p-2}{k_1-1}\sum_{a=1}^{k_1-1}U_{\Gamma_{k_1,k_1,s},n-p-k_1-a}(y)\right). \end{aligned}$$

Thus we have the following theorem.

Theorem 6. *Let* $\Gamma_{k_1,k_1,s} = \Gamma_{k_1,k_1} \cup \{12\cdots s(s+1)\}$ *where* $s \geq k_1$. *Then* $U_{\Gamma_{k_1,k_1,s},1}(y) =$

$-y$ *and for* $n \geq 2$,

$$U_{\Gamma_{k_1,k_1,s},n}(y) = \\ -y\sum_{r=1}^{k_1-1} U_{\Gamma_{k_1,k_1,s},n-r}(y) - y\left(\sum_{p=0}^{s-k_1}\left(1+\binom{n-p-2}{k_1-1}\right)U_{\Gamma_{k_1,k_1,s},n-p-k_1}(y)\right. \\ \left. + y\binom{n-p-2}{k_1-1}\sum_{a=1}^{k_1-1} U_{\Gamma_{k_1,k_1,s},n-p-k_1-a}(y)\right).$$

For example, if $k_1 = 2$, then

$$U_{\Gamma_{2,2,s},n}(y) = -yU_{\Gamma_{2,2,s},n-1}(y) \\ -y\left(\sum_{p=0}^{s-2}(n-p-1)U_{\Gamma_{2,2,s},n-2-p}(y) + (n-p-2)yU_{\Gamma_{2,2,s},n-3-p}(y)\right).$$

We shall further explore two special cases, namely, $k_1 = k_2 = s = 2$ where the recursion becomes

$$U_{\Gamma_{2,2,2},n}(y) = -yU_{\Gamma_{2,2,2},n-1}(y) - y(n-1)U_{\Gamma_{2,2,2},n-2}(y) - y^2(n-2)U_{\Gamma_{2,2,2},n-3}(y) \quad (16)$$

for $n > 1$, and $k_1 = k_2 = 2, s = 3$ where the recursion becomes

$$U_{\Gamma_{2,2,3},n}(y) = -yU_{\Gamma_{2,2,3},n-1}(y) - y(n-1)U_{\Gamma_{2,2,3},n-2}(y) - y^2(n-2)U_{\Gamma_{2,2,3},n-3}(y) - \\ y(n-2)U_{\Gamma_{2,2,3},n-3}(y) - y^2(n-3)U_{\Gamma_{2,2,3},n-4}(y). \quad (17)$$

Tables 3 and 4 below give the polynomials $U_{\Gamma_{2,2,2},n}(-y)$ for even and odd values of n, respectively.

k	n	$U_{\Gamma_{2,2,2},2k}(-y)$
1	2	$y + y^2$
2	4	$3y^2 + 3y^3 + y^4$
3	6	$15y^3 + 15y^4 + 5y^5 + y^6$
4	8	$105y^4 + 105y^5 + 35y^6 + 7y^7 + y^8$
5	10	$945y^5 + 945y^6 + 315y^7 + 63y^8 + 9y^9 + y^{10}$
6	12	$10395y^6 + 10395y^7 + 3465y^8 + 693y^9 + 99y^{10} + 11y^{11} + y^{12}$
7	14	$135135y^7 + 135135y^8 + 45045y^9 + 9009y^{10} + 1287y^{11} + 143y^{12} + 13y^{13} + y^{14}$

Table 3: The polynomials $U_{\Gamma_{2,2,2},2k}(-y)$ for $\Gamma_{2,2,2} = \{1324, 1423, 123\}$

This data leads us to conjecture the following explicit formulas:

$$U_{\Gamma_{2,2,2},2k}(-y) = \sum_{i=0}^{k}(2k-1)\downarrow\downarrow_{k-i} y^{k+i} \quad (18)$$

$$U_{\Gamma_{2,2,2},2k+1}(-y) = \sum_{i=0}^{k}(2k)\downarrow\downarrow_{k-i} y^{k+1+i} \quad (19)$$

k	n	$U_{\Gamma_{2,2,2},2k+1}(-y)$
1	3	$2y^2+y^3$
2	5	$8y^3+4y^4+y^5$
3	7	$48y^4+24y^5+6y^6+y^7$
4	9	$384y^5+192y^6+48y^7+8y^8+y^9$
5	11	$3840y^6+1920y^7+480y^8+80y^9+10y^{10}+y^{11}$
6	13	$46080y^7+23040^8+5760^9+960y^{10}+120y^{11}+12y^{12}+y^{13}$
7	15	$645120y^8+322560y^9+80640y^{10}+13440y^{11}+1680y^{12}+168y^{13}+14y^{14}+y^{15}$

Table 4: The polynomials $U_{\Gamma_{2,2,2},2k+1}(-y)$ for $\Gamma_{2,2,2}=\{1324,1423,123\}$

where $(x)\downarrow\downarrow_0=1$ and $(x)\downarrow\downarrow_k=x(x-2)(x-4)\cdots(x-2k-2)$ for $k\geq 1$.

These formulas can be proved by induction. Note that it follows from (16) that for $n>1$,

$$U_{\Gamma_{2,2,2},n}(-y)=yU_{\Gamma_{2,2,2},n-1}(-y)+y(n-1)U_{\Gamma_{2,2,2},n-2}(-y)-y^2(n-2)U_{\Gamma_{2,2,2},n-3}(-y). \tag{20}$$

One can directly check these formulas for $n\leq 3$. For $n>3$, let $U_{\Gamma_{2,2,2},n}(-y)|_{y^k}$ be the coefficient of y^k in $U_{\Gamma_{2,2,2},n}(-y)$. Equation (20) allows us to write the coefficient of y^{k+1+i}, for $0\leq i\leq k$, in $U_{\Gamma_{2,2,2},2k+1}(-y)$ as

$$\begin{aligned}U_{\Gamma_{2,2,2},2k+1}(-y)|_{y^{k+1+i}} &= U_{\Gamma_{2,2,2},2k}(-y)|_{y^{k+i}}+(2k)U_{\Gamma_{2,2,2},2k-1}(-y)|_{y^{k+i}}\\ &\quad -(2k-1)U_{\Gamma_{2,2,2},2k-2}(-y)|_{y^{k+i-1}}\\ &= (2k-1)\downarrow\downarrow_{k-i}+(2k)(2k-2)\downarrow\downarrow_{k-i}-(2k-1)\cdot(2k-3)\downarrow\downarrow_{k-i}\\ &= (2k)\downarrow\downarrow_{k-i}.\end{aligned}$$

For the even case when $n=2k$, the coefficient of y^{k+i}, for $0\leq i\leq k$, in $U_{\Gamma_{2,2,2},2k}(-y)$ is

$$\begin{aligned}U_{\Gamma_{2,2,2},2k}(-y)|_{y^{k+i}} &= U_{\Gamma_{2,2,2},2k-1}(-y)|_{y^{k+i-1}}+(2k-1)U_{\Gamma_{2,2,2},2k-2}(-y)|_{y^{k+i-1}}\\ &\quad -(2k-2)U_{\Gamma_{2,2,2},2k-3}(-y)|_{y^{k+i-2}}\\ &= (2k-2)\downarrow\downarrow_{k-i}+(2k-1)(2k-3)\downarrow\downarrow_{k-i}-(2k-2)\cdot(2k-4)\downarrow\downarrow_{k-i}\\ &= (2k-1)\downarrow\downarrow_{k-i}.\end{aligned}$$

This proves equations (18) and (19).

Hence, we can give a closed formula for $\mathrm{NM}_{\Gamma_{2,2,2}}(t,x,y)$. That is, we have the following theorem.

Theorem 7.

$$NM_{\Gamma_{2,2,2}}(t,x,y)=$$
$$\left(\frac{1}{1+\left(\sum_{n\geq 1}\frac{t^n}{n!}\sum_{i=0}^{k}(2k-1)\downarrow\downarrow_{k-i}y^{k+i}\right)+\left(\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i=0}^{k}(2k)\downarrow\downarrow_{k-i}y^{k+1+i}\right)}\right)^x.$$

It follows from (17) that

$$U_{\Gamma_{2,2,3},n}(-y) = yU_{\Gamma_{2,2,3},n-1}(-y) + y(n-1)U_{\Gamma_{2,2,3},n-2}(-y) + y(n-2)U_{\Gamma,n-3}(-y) \\ - y^2(n-2)U_{\Gamma_{2,2,3},n-3}(-y) - y^2(n-3)U_{\Gamma_{2,2,3},n-4}(-y).$$

The next three tables below give the polynomials $U_{\Gamma_{2,2,3},n}(y)$ for $n = 3k, n = 3k+1$, and $n = 3k+2$, respectively.

k	n	$U_{\Gamma_{2,2,3},3k}(-y)$
1	3	$y + 2y^2 + y^3$
2	6	$4y^2 + 33y^3 + 19y^4 + 5y^5 + y^6$
3	9	$28y^3 + 767y^4 + 781y^5 + 267y^6 + 55y^7 + 8y^8 + y^9$
4	12	$280y^4 + 20496y^5 + 44341y^6 + 20765y^7 + 5137y^8 + 861y^9$ $+109y^{10} + 11y^{11} + y^{12}$
5	15	$3640y^5 + 598892y^6 + 2825491y^7 + 2072739y^8 + 641551y^9 + 125111y^{10}$ $+17755y^{11} + 1977y^{12}181y^{13} + 14y^{14} + y^{15}$

Table 5: The polynomials $U_{\Gamma_{2,2,3},3k}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$

k	n	$U_{\Gamma_{2,2,3},3k+1}(-y)$
1	4	$5y^2 + 3y^3 + y^4$
2	7	$67y^3 + 81y^4 + 29y^5 + 6y^6 + y^7$
3	10	$1166y^4 + 3321y^5 + 1645y^6 + 417y^7 + 71y^8 + 9y^9 + y^{10}$
4	13	$23746y^5 + 160647y^6 + 128771y^7 + 41055y^8 + 8137y^9 + 1167y^{10}$ $+131y^{11} + 12y^{12} + y^{13}$
5	16	$550844y^6 + 8107518y^7 + 12109429y^8 + 5170965y^9 + 1225973y^{10}$ $+200253y^{11} + 24889y^{12} + 2493y^{13} + 209y^{14} + 15y^{15} + y^{16}$

Table 6: The polynomials $U_{\Gamma_{2,2,3},3k+1}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$

k	n	$U_{\Gamma_{2,2,3},3k+2}(-y)$
1	5	$7y^2 + 11y^3 + 4y^4 + y^5$
2	8	$70y^3 + 297y^4 + 157y^5 + 41y^6 + 7y^7 + y^8$
3	11	$910y^4 + 10343y^5 + 9223y^6 + 3069y^7 + 613y^8 + 89y^9 + 10y^{10} + y^{11}$
4	14	$14560y^5 + 390564y^6 + 687109y^7 + 306413y^8 + 74137y^9 + 12261y^{10}$ $+1537y^{11} + 155y^{12} + 13y^{13} + y^{14}$

Table 7: The polynomials $U_{\Gamma_{2,2,3},3k+2}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$

For any $s \geq 3$, it is easy to see that the lowest power of y that occurs in $U_{\Gamma_{2,2,s},n}(-y)$ corresponds to brick tabloids where we use the minimum number of bricks. Since the maximum size of brick in a fixed point of $I_{\Gamma_{2,2,s}}$ is s, we see that the minimum number of bricks that we can use for a fixed point of $I_{\Gamma_{2,2,s}}$ of length sn is n while the minimum number of bricks that we can use for a fixed point of $I_{\Gamma_{2,2,s}}$ of length $sn+j$ for $1 \leq j \leq s-1$ is $n+1$. We can prove the following general theorem for the coefficients of the lowest power of y that appears in $U_{\Gamma_{2,2,s},n}(-y)$.

Theorem 8. *For $n \geq 1$,*

$$U_{\Gamma_{2,2,s},sn}(-y)|_{y^n} = \prod_{i=1}^{n}((i-1)s+1) \tag{21}$$

and

$$U_{\Gamma_{2,2,s},sn+s-1}(-y)|_{y^{n+1}} = \prod_{i=1}^{n}((i+1)s+1). \tag{22}$$

Proof. For (21), we first notice that any fixed point (B,σ) of $I_{\Gamma_{2,2,s}}$ that contributes to $U_{\Gamma_{2,2,s},sn}(-y)|_{y^n}$ must have only bricks of size s. Thus $B = (s,\ldots,s)$. We shall prove (21) by induction on n. Clearly, $U_{\Gamma_{2,2,s},s}(-y)|_y = 1$. Now suppose (B,σ) is a fixed point of $I_{\Gamma_{2,2,s}}$ of size sn where $\sigma = \sigma_1 \cdots \sigma_{sn}$. By our arguments above, the first $s-1$ elements of the first brick must be $1,2,\ldots,s-1$, reading from left to right. The element in the next cell σ_s can be arbitrary. That is, if it is equal to s, then there will be an increase between the first two bricks and if $\sigma_s > s$, then it must be the case that $\sigma_{s+1} = s$ in which case there will by $\Gamma_{2,2,s}$-match that involves the last two cells of the first brick and the first two cells of the next brick. We can then remove the first brick and adjust the remaining numbers to produce a fixed point O' of $I_{\Gamma_{2,2,s}}$ of length $s(n-1)$ in which every brick is of size s. It follows by induction that

$$\begin{aligned} U_{\Gamma_{2,2,s},sn}(-y)|_{y^n} &= ((n-1)s+1)U_{\Gamma_{2,2,s},s(n-1)}(-y)|_{y^{n-1}} \\ &= ((n-1)s+1)\prod_{i=1}^{n-1}((i-1)s+1) \\ &= \prod_{i=1}^{n}((i-1)s+1). \end{aligned}$$

Next consider $U_{\Gamma_{2,2,s},2s-1}(-y)|_{y^2}$. In this case, either the first brick of size $s-1$ or the first brick is of size s. If the first brick is of size s, then we can argue as above that the first $s-1$ elements of the first brick are $1,\ldots,s-1$, and we have s choices for the last element of the first brick. If the first brick is of size $s-1$, then we can argue as above that the first $s-2$ elements of the first brick are $1,\ldots,s-2$, and we have $s+1$ choices for the last element of the first brick. Thus

$$U_{\Gamma_{2,2,s},2s-1}(-y)|_{y^2} = 2s+1.$$

Next consider $U_{\Gamma_{2,2,s},(ns+s-1)}(-y)|_{y^{n+1}}$. In such a situation, any fixed point (B,σ) of $I_{\Gamma_{2,2,s}}$ that can contribute to $U_{\Gamma_{2,2,s},(ns+s-1)}(-y)|_{y^{n+1}}$ must have n bricks of size s

and one brick of size $s-1$. If the first brick is of size s, then we can argue as above that the first $s-1$ elements of the first brick are $1,\ldots,s-1$, and we have sn choices for the last element of the first brick. Then we can remove this first brick and adjust the remaining numbers to produce a fixed point O' in $I_{\Gamma_{2,2,s}}$ of size $(n-1)s+s-1$ which has $n-1$ bricks of size s and one brick of size $s-1$. If the first brick is of size $s-1$, then we can argue as above that the first $s-2$ elements of the first brick are $1,\ldots,s-2$, and we have $sn+1$ choices for the last element of the first brick. Then we can remove this first brick and adjust the remaining numbers to produce a fixed point O' in $I_{\Gamma_{2,2,s}}$ of size ns which has n bricks of size s

Thus if $n \geq 2$,

$$\begin{aligned}
U_{\Gamma_{2,2,s},(ns+s-1)}(-y)|_{y^{n+1}} &= (sn+1)U_{\Gamma_{2,2,s},ns}(-y)|_{y^n} + (sn)U_{\Gamma_{2,2,s},((n-1)s+s-1)}(-y)|_{y^n} \\
&= (sn+1)\prod_{i=1}^{n}((i-1)s+1) + (sn)\prod_{i=1}^{n-1}((i+1)s+1) \\
&= (s+1)\prod_{i=1}^{n-1}((i+1)s+1) + (sn)\prod_{i=1}^{n-1}((i+1)s+1) \\
&= ((n+1)s+1)\prod_{i=1}^{n-1}((i+1)s+1) \\
&= \prod_{i=1}^{n}((i+1)s+1).
\end{aligned}$$

□

Unfortunately, we cannot extend this type of argument to compute $U_{\Gamma_{2,2,s},ns+k}(-y)|_{y^{n+1}}$ where $1 \leq k \leq s-2$. The problem is that we have more than one choice for the sizes of the bricks in such cases. For example, to compute $U_{\Gamma_{2,2,3},4}(-y)|_{y^3}$, the brick sizes could be some rearrangement of (3,1) or (2,2). One can use our recursions to compute $U_{\Gamma_{2,2,s},ns+k}(-y)|_{y^{n+1}}$ for small values of s. For example, we can find all the coefficients of the lowest power of $U_{\Gamma_{2,2,3},n}(-y)$. That is, we claim

(i) $U_{\Gamma_{2,2,3},3k}(-y)|_{y^k} = \prod_{i=1}^{k}(3(i-1)+1)$,

(ii) $U_{\Gamma_{2,2,3},3k+2}(-y)|_{y^{k+1}} = \prod_{i=1}^{k}(3(i+1)+1)$, and

(iii) if $A_k = U_{\Gamma,3k+1}(-y)|_{y^{k+1}}$ then $A_1 = 5$ and $A_k = (3k-1)A_{k-1}+(3k)\prod_{i=1}^{k-1}(3i+4)$ for all $k \geq 2$.

Clearly, (i) and (ii) follow from our previous theorem. To prove (iii), note that

$$\begin{aligned}
A_k = U_{\Gamma,3k+1}(-y)|_{y^{k+1}} &= U_{\Gamma,3k}(-y)|_{y^k} + (3k)U_{\Gamma,3k-1}(-y)|_{y^k} + (3k-1)U_{\Gamma,3k-2}(-y)|_{y^k} \\
&\quad - (3k-1)U_{\Gamma,3k-2}(-y)|_{y^{k-1}} - (3k-2)U_{\Gamma,3k-3}(-y)|_{y^{k-1}} \\
&= \prod_{i=1}^{k}(3i-2) + (3k)\prod_{i=1}^{k-1}(3i+4) + (3k-1)U_{\Gamma,3k-2}(-y)|_{y^k} \\
&\quad - (3k-2)\prod_{i=1}^{k-1}(3i-2) \\
&= (3k)\prod_{i=1}^{k-1}(3i+4) + (3k-1)U_{\Gamma,3k-2}(-y)|_{y^k} \\
&= (3k-1)A_{k-1} + (3k)\prod_{i=1}^{k-1}(3i+4).
\end{aligned}$$

This explains all the coefficients for the smallest power of y in the polynomials $U_{\Gamma_{2,2,3},n}(-y)$ for the family $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$.

5 The Proofs of Theorem 1 and Theorem 2

In this section, we will study two more examples of the differences between the recursions for $U_{\Gamma,n}(y)$'s and the recursions for $U_{\Gamma\cup\{12\cdots s(s+1)\},n}(y)$'s. In particular, we will prove Theorems 1 and 2.

Proof of Theorem 1

Let $\Gamma = \{1324, 123\}$. Let (B,σ) be a fixed point I_Γ where $B = (b_1,\ldots,b_k)$ and $\sigma = \sigma_1\cdots\sigma_n$. By Lemma 5, we know that all the bricks b_i must be of size 1 or 2. Since the minimal elements in bricks of B must weakly increase, we see that 1 must be in cell 1 and 2 must be either in b_1 or it is in the first cell of b_2. Thus we have three possibilities.

Case 1. 2 is in b_1.

In this case, b_1 must be of size 2 and we can remove b_1 from (B,σ) are reduce the remaining numbers by 2 to get a fixed point of I_Γ of size $n-2$. It then easily follows that the fixed points in Case 1 contribute $-yU_{\Gamma,n-2}(y)$ to $U_{\Gamma,n}(y)$.

Case 2. 2 is in b_2 and $b_1 = 1$.

In this case, it is easy to see that 1 cannot be involved in any Γ-match so that we can remove b_1 from (B,σ) are reduce the remaining numbers by 1 to get a fixed point of I_Γ of size $n-1$. It follows that the fixed points in Case 2 contribute $-yU_{\Gamma,n-1}(y)$ to $U_{\Gamma,n}(y)$.

Case 3. 2 is in b_2 and $b_1 = 2$.

In this case, there is descent between bricks b_1 and b_2 so that there must be a 1324-match in σ contained in the cells of b_1 and b_2. In particular, this means $b_2 = 2$ and there is 1324-match starting at 1 in σ. We then have two subcases.

Subcase 3.a. There is no 1324-match in (B, σ) starting at cell 3

We claim that $\{\sigma_1, \ldots, \sigma_4\} = \{1, 2, 3, 4\}$. If not, let $d = \min(\{1, 2, 3, 4\} - \{\sigma_1, \ldots, \sigma_4\})$. Then d must be in cell 5, the first cell of brick b_3 and there is a decrease between bricks b_2 and b_3 since $d \leq 4 < \sigma_4$. Thus, in order to avoid combining bricks b_2 and b_3, we need a 1324-match among the cells of these two bricks. However, the only possible 1324-match among the cells of b_2 and b_3 would have to start at cell 3 where $\sigma_3 = 2$. This contradicts the assumption that there is no 1324-match in (B, σ) starting at cell 3. As a result, it must be the case that the first four numbers must occupy the first four cells of (B, σ) so we must have $\sigma_1 = 1$, $\sigma_2 = 3$, $\sigma_3 = 2$, $\sigma_4 = 4$, and $\sigma_5 = 5$. It then follows that if we let O' be the result by removing the first four cells from (B, σ) and then subtract 4 from the remaining entries in the cells, then O' will be a fixed point in $\mathcal{O}_{\Gamma,n-4}$. It then easily follows that the contribution of fixed points in subcase 3.a to $U_{\Gamma,n}(y)$ is $(-y)^2 U_{\Gamma,n-4}(y)$.

Subcase 3.b. There is a 1324-match in O starting at cell 3

In this case, there is decrease between bricks b_2 and b_3. Hence, the 1324-match starting at cell 3 must be contained in the cells of b_2 and b_3 so that b_3 must be of size 2. In general, suppose that the bricks $b_2, \ldots, b_{k-1}$ all have exactly two cells and there are 1324-matches starting at cells $1, 3, \ldots, 2k-3$ but there is no 1324-match starting at cell $2k-1$ in O.

Similar to Subcase 3.a, we will show that $\{\sigma_1, \ldots, \sigma_{2k}\} = \{1, 2, \ldots, 2k\}$. That is, the first $2k$ numbers must occupy the first $2k$ cells in O. If not, let $d = \min(\{1, 2, \ldots, 2k\} - \{\sigma_1, \ldots, \sigma_{2k}\})$. Since the minimal elements of the bricks are weakly increasing, it must be the case that d is in the first cell of b_{k+1}. Next, the fact that there are 1324-matches starting in cells $1, 3, \ldots, 2k-1$ easily implies that σ_{2k} is the largest element in $\{\sigma_1, \ldots, \sigma_{2k}\}$ which means that $\sigma_{2k} > d$. But then there is a decrease between bricks b_k and b_{k+1} which means that there must be a 1324-match contained in the cells of b_k and b_{k+1}. This implies that there is a 1324-match starting at cell $2k-1$ which contradicts our assumption.

Thus, if we remove the first $2k$ cells of (B, σ) and subtract $2k$ from the remaining elements, we will obtain a fixed point O' in $\mathcal{O}_{\Gamma,n-2k}$. Therefore, each fixed point O in this case will contribute $(-y)^k U_{\Gamma,n-2k}(y)$ to $U_{\Gamma,n}(y)$. The final task is to count the number of permutations $\sigma_1 \cdots \sigma_{2k}$ of $\mathfrak{S}_{2k}$ that has 1324-matches starting at positions $1, 3, \ldots, 2k-3$. In [13], Jones and Remmel gave a bijection between the set of such σ and the set of Dyck paths of length $2k-2$. Hence, there are C_{k-1} such fixed points, where $C_n = \frac{1}{n-1}\binom{2n}{n}$ is the n^{th} Catalan number. It then easily follows that the contribution of the fixed points in Subcase 3.b to $U_{\Gamma,n}(y)$ is

$$\sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^k C_{k-1} U_{\Gamma,n-2k}(y).$$

Hence, we know that $U_{\Gamma,1} = -y$ and for $n > 1$,

$$U_{\Gamma,n}(y) = -yU_{\Gamma,n-1}(y) - yU_{\Gamma,n-2}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^k C_{k-1} U_{\Gamma,n-2k}(y).$$

This proves Theorem 1.

We have computed the polynomials $U_{\{1324,123\},n}(-y)$ for small n which are given in the Table 8 below.

n	$U_{\{1324,123\},n}(-y)$
1	y
2	$y + y^2$
3	$2y^2 + y^3$
4	$2y^2 + 3y^3 + y^4$
5	$5y^3 + 4y^4 + y^5$
6	$5y^3 + 9y^4 + 5y^5 + y^6$
7	$14y^4 + 14y^5 + 6y^6 + y^7$
8	$14y^4 + 28y^5 + 20y^6 + 7y^7 + y^8$
9	$42y^5 + 48y^6 + 27y^7 + 8y^8 + y^9$
10	$42y^5 + 90y^6 + 75y^7 + 35y^8 + 9y^9 + y^{10}$

Table 8: The polynomials $U_{\Gamma,n}(-y)$ for $\Gamma = \{1324, 123\}$

An anonymous referee observed that up to a power of y, the odd rows are the triangle A039598 in the OEIS and the even rows are the triangle A039599 in the OEIS. These tables arise in expanding the powers of x in terms of the Chebyshev polynomials of the second kind. Since there are explicit formula for entries in these tables, we have the following theorem.

Theorem 9. *Let* $\Gamma = \{1324, 123\}$. *Then for all* $n \geq 0$,

$$U_{\Gamma,2n}(y) = \sum_{k=0}^{n} \frac{(2k+1)\binom{2n}{n-k}}{n+k+1}(-y)^{n+k+1} \tag{23}$$

and

$$U_{\Gamma,2n+1}(y) = \sum_{k=0}^{n} \frac{2(k+1)\binom{2n+1}{n-k}}{n+k+2}(-y)^{n+k} \tag{24}$$

Proof. First we consider the polynomials $U_{\Gamma,2n+1}(-y)$ which correspond to the entries in the table $T(n,k)$ for $0 \leq k \leq n$ of entry A039598 in the OEIS. $T(n,k)$ has an explicit formula, namely,

$$T(n,k) = \frac{2(k+1)\binom{2n+1}{n-k}}{n+k+2}$$

for all $n \geq 0$ and $0 \leq k \leq n$. Let $\mathcal{T}(n,k)$ be set all of paths of length $2n+1$ consisting of either up steps $(1,1)$ or down steps $(1,-1)$ that start at (0,0) and end at $(2n + 1, 2k + 1)$ which stay above the x-axis. Then one of the combinatorial interpretations of the $T(n, k)$'s is that $T(n, k) = |\mathcal{T}(n, k)|$. Let $\mathcal{F}_{2n+1,2k+1}$ be the set of all fixed points of I_Γ with $2k+1$ bricks of size 1 and $n-k$ bricks of size 2. We will construct a bijection $\theta_{n,k}$ from $\mathcal{F}_{2n+1,2k+1}$ onto $\mathcal{T}(n, k)$. Note all $(B, \sigma) \in \mathcal{F}_{2n+1,2k+1}$ have weight $(-y)^{n+k+1}$ so that the bijections $\theta_{n,k}$ will prove (24).

First we must examine the fixed points of I_Γ in greater detail. Note that since Γ contains the identity permutation 123, all the bricks in any fixed point of I_Γ must be of size 1 or size 2. Next, we consider the structure of the fixed points of I_Γ which have k bricks of size 1 and ℓ bricks of size 2. Suppose (B, σ) is such a fixed point where $B = (b_1, \ldots, b_{k+\ell})$ and that the bricks of size 1 in B are $b_{i_1}, \ldots, b_{i_k}$ where $1 \leq i_1 < \cdots < i_k \leq k + \ell$. For any s, there cannot be a decrease between brick b_{i_j-1} and brick b_{i_j} in B since otherwise we could combine bricks b_{i_j-1} and b_{i_j}, which would violate our assumption that (B, σ) is a fixed point of I_Γ. Next we claim that if there are s bricks of size 2 that come before brick b_{i_j} so that b_{i_j} covers cell $2s + j$ in (B, σ), then $\sigma_{2s+j} = 2s + j$ and $\{\sigma_1, \ldots, \sigma_{2s+j}\} = \{1, \ldots, 2s + j\}$. To prove this claim, we proceed by induction. For the base case, suppose that b_{i_1} covers cell $2s+1$ so that (B, σ) starts out with s bricks of size 2. If $s = 0$, there is nothing to prove. Next suppose that $s = 1$. Then we know that in all fixed points of I_Γ, 2 must be in cell 2 or cell 3. Since there is an increase between b_1 and b_2, it must be the case that 1 and 2 lie in b_1 and since the minimal elements in the brick form a weakly increasing sequence, it must be the case that b_2 is filled with 3. If $s \geq 2$, then for $1 \leq i < s$, either there is an increase between b_i and b_{i+1} in which case the elements in b_i and b_{i+1} must match the pattern 1234, or there is a decrease between b_i and b_{i+1} in which case the four elements must match the pattern 1324. This means that if for each brick of size 2, we place the second element of the brick on the top of the first element, then any two consecutive bricks will be one of the two forms pictured in Figure 4. Thus if we consider the $s \times 2$ array built from the first s bricks of size 2, we will obtain a column strict tableaux with distinct entries of shape (s, s). In particular, it must be the case that the largest element in the array is the element which appears at the top of the last column. That element corresponds to the second cell of brick b_s. Since there is an increase between brick b_s and brick b_{s+1} it must mean that the element in brick b_{s+1} is larger than any of the elements that appear in bricks $b_1, \ldots, b_s$. Thus $\sigma_i < \sigma_{2s+1}$ for $i \leq 2s$. Since the minimal elements in the bricks are increasing, it follows that $\sigma_{2s+1} < \sigma_j$ for all $j > 2s + 1$ so that it must be the case that $\sigma_{2s+1} = 2s + 1$ and $\{\sigma_1, \ldots, \sigma_{2s+1}\} = \{1, \ldots, 2s + 1\}$. Thus the base case of our induction holds.

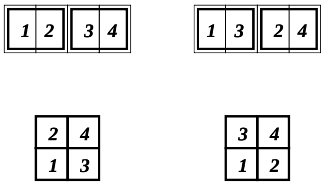

Figure 4: Patterns for two consecutive brick of size 2 in a fixed point of I_Γ.

We can repeat the same argument for i_j where $j > 1$. That is, by induction, we can assume that if there are r bricks of size 2 that precede brick $b_{i_{j-1}}$, then $\sigma_{2r+j-1} = 2r + j - 1$ and $\{\sigma_1, \ldots, \sigma_{2r+j-1}\} = \{1, \ldots, 2r + j - 1\}$. Hence if we remove these elements and subtract $2r + j - 1$ from the remaining elements in (B, σ), we would end up with a fixed point of I_γ. Thus we can repeat our argument for the base case to prove that if there are s bricks of size 2 between brick $b_{i_{j-1}}$ and b_{i_j}, then $\sigma_{2r+2s+j} = 2r + 2s + j$ and $\{\sigma_1, \ldots, \sigma_{2r+2s+j}\} = \{1, \ldots, 2r + 2s + j\}$.

Next we note that there is a well known bijection ϕ between standard tableaux of shape (n, n) and Dyck paths of length $2n$, see [19]. Here a Dyck path is path consisting of either up steps $(1, 1)$ or down steps $(1, -1)$ that starts at (0,0) and ends at $(2n, 0)$ which stays above the x-axis. Given a standard tableau T, $\phi(T)$ is the Dyck path whose i-th segment is an up step if i is the first row and whose i-th segment is a down step if i is in the second row. This bijection is illustrated in Figure 5.

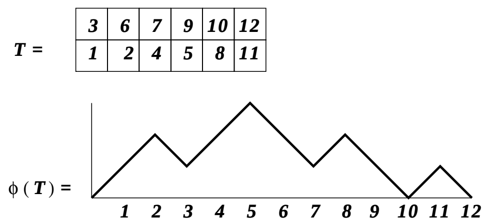

Figure 5: The bijection ϕ.

We can now easily describe our desired bijection $\theta_{n,k}$. Starting with a fixed point (B, σ) in $\mathcal{F}_{2n+1,2k+1}$ where $B = (b_1, \ldots, b_{n+k+1})$, we can rotate all the bricks of size 2 by -90 degrees and end up with an array consisting of bricks of size one and $2 \times r$ arrays corresponding to standard tableaux. For example, this step is pictured in the second row of Figure 6. By our remarks above, each $2 \times r$ array corresponds to standard tableaux of shape (r, r) where the entries lie in some consecutive sequence of elements from $\{1, \ldots, 2n + 1\}$. Suppose that $b_{i_1}, \ldots, b_{i_{2k+1}}$ are the bricks of size 1 in B where $i_1 < \cdots < i_{2k+1}$. Let T_j be the standard tableau corresponding to the consecutive string of brick of size 2 immediately preceding brick b_{i_j} and P_i be the Dyck path $\phi(T_i)$. If there is no bricks of size 2 immediately preceding b_{i_j}, then P_j is just the empty path. Finally let T_{2k+2} the standard tableau corresponding to the bricks of size 2 following $b_{i_{2k+1}}$ and P_{2k+2} be the Dyck path corresponding to $\phi(T_{2k+2})$ where again P_{2k+2} is the empty path if there are no bricks of size 2 following $b_{i_{2k+1}}$. Then

$$\theta_{n,k}(B, \sigma) = P_1(1, 1)P_2(1, 1) \ldots P_{2k+1}(1, 1)P_{2k+2}.$$

For example, line 3 of Figure 6 illustrates this process. In fact, it easy to see that if i is in the bottom row of intermediate diagram for (B, σ), then the i-th segment of $\theta_{n,k}(B, \sigma)$ is an up step and if i is in the top row of intermediate diagram for (B, σ), then the i-th segment of $\theta_{n,k}(B, \sigma)$ is an down step.

The inverse of $\theta_{n,k}$ is also easy to describe. That is, given a path P in $\mathcal{T}(n, k)$, we let d_i be the step that corresponds to the last up step that ends at level i. Then P can be factored as

$$P_1 d_1 P_2 d_2 \ldots P_{2k+1} d_{2k+1} P_{2k+2}$$

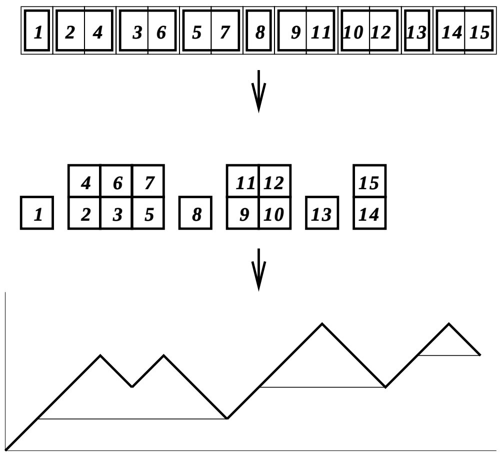

Figure 6: The bijection $\theta_{n,k}$.

where each P_i is a path that corresponds to a Dyck path that starts at level $i-1$ and ends at level $i-1$ and stays above the line $x = i-1$. Then for each i, $T_i = \phi^{-1}(P_i)$ is a standard tableau. Using these tableaux and being cognizant of the restrictions on the initial segments of elements of $\mathcal{F}_{2n+1,2k+1}$ preceding bricks of size 1, one can easily reconstruct the 2 line intermediate array corresponding to $T_1 d_1 T_2 d_2 \ldots T_{2k+1} d_{2k+1} T_{2k+2}$. For example, this process is pictured on line 2 of Figure 7. Then we only have to rotate all the bricks of size corresponding to a bricks of height 2 by 90 degrees to obtain $\theta_{n,k}^{-1}(P)$. This step is pictured on line 3 of Figure 7.

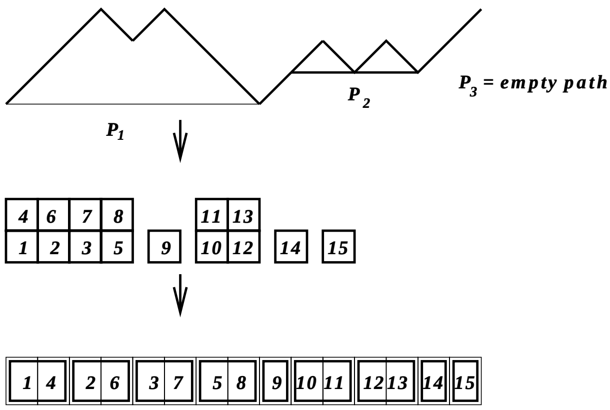

Figure 7: The bijection $\theta_{n,k}^{-1}$.

Next we consider the polynomials $U_{\Gamma,2n}(-y)$ which correspond to the entries in the table $R(n,k)$ for $0 \leq k \leq n$ of entry A039599 in the OEIS. $R(n,k)$ has an explicit formula, namely,

$$R(n,k) = \frac{(2k+1)\binom{2n}{n-k}}{n+k+1}$$

for all $n \geq 0$ and $0 \leq k \leq n$. Let $\mathcal{R}(n,k)$ be set all of paths of length $2n$ consisting of either up steps $(1,1)$ or down steps $(1,-1)$ that start at (0,0) and end at $(2n,0)$

that have k down steps that end on the line $x = 0$. Here there is no requirement that the paths stay above the x-axis. Then one of the combinatorial interpretations of the $R(n,k)$s is that $R(n,k) = |\mathcal{R}(n,k)|$. Let $\mathcal{F}_{2n,2k}$ be the set of all fixed points of I_Γ with $2k$ bricks of size 1 and $n-k$ bricks of size 2. We will construct a bijection $\beta_{n,k}$ from $\mathcal{F}_{2n,2k}$ onto $\mathcal{R}(n,k)$. Note all $(B,\sigma) \in \mathcal{F}_{2n,2k}$ weight $(-y)^{n+k}$ so that the bijections $\beta_{n,k}$ will prove (24).

We can now easily describe our desired bijection $\beta_{n,k}$. Starting with a fixed point (B,σ) in $\mathcal{F}_{2n,2k1}$ where $B = (b_1, \ldots, b_{n+k})$, we can rotate all the bricks of size 2 by -90 degrees and end up with an array consisting of bricks of size one and $2 \times r$ arrays corresponding to standard tableaux. For example, this step is pictured in the second row of Figure 9. By our remarks above, each $2 \times r$ array corresponds to standard tableaux of shape (r,r) where the entries lie in some consecutive sequence of elements from $\{1, \ldots, 2n\}$. Suppose that $b_{i_1}, \ldots, b_{i_{2k}}$ are the bricks of size 1 in B where $i_1 < \cdots < i_{2k}$. Let T_s be the standard tableau corresponding to the bricks of size 2 immediately preceding brick b_{j_s} for $1 \leq s \leq 2n$ and let T_{2k+1} be the standard tableau corresponding to the bricks of size 2 following brick $b_{i_{2k}}$. For $i = 0, \ldots, 2k+1$, let P_i be the Dyck path $\phi(T_i)$. In each case j where there are no such bricks of size 2, then P_j is just the empty path. For each such i, let $\overline{P}_i$ denote the flip of P_i, i.e. the path that is obtained by flipping P_i about the x-axis. For example, the process of flipping a Dyck path is pictured in Figure 8.

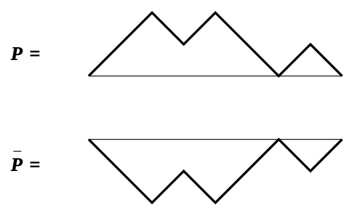

Figure 8: The flip of Dyck path.

Then

$$\beta_{n,k}(B,\sigma) = \overline{P}_1(1,1)P_2(1,-1)\overline{P}_3(1,1)P_4(1,-1)\ldots\overline{P}_{2k-1}(1,1)P_{2k}(1,-1)\overline{P}_{2k+1}.$$

That is, each pair $b_{i_{2j-1}}, b_{i_{2j}}$ will correspond to an up step starting at $x = 0$ followed by a Dyck path which starts at ends a line $x = 1$ followed by down step ending at $x = 0$. These segments are then connected by flips of Dyck path that stay below the x-axis. Thus $\beta_{n,k}(B,\sigma)$ will have exactly k down steps that end at $x = 0$. For example, line 3 of Figure 9 illustrates this process.

The inverse of $\beta_{n,k}$ is also easy to describe. That is, given a path P in $\mathcal{R}(n,k)$, let $f_1, \ldots, f_k$ be the positions of the down steps that end at $x = 0$ and define $e_1, \ldots, e_k$ so that e_1 is the right most up step that starts at $x = 0$ and precedes f_1 and for $2 \leq i \leq k$,, e_i is the right most up step that follows f_{i-1} and precedes f_i. It is then easy to see that the path Q_1 which precedes e_1 must be a path that starts at (0,0) and ends at $(e_1 - 1, 0)$ and stays below the x-axis so that Q_1 is the flip of some Dyck path P_1. Next, the path Q_2 between $(e_1, 1)$ and $(f_1 - 1, 1)$ must either be empty or is a path which stays above the line $x = 1$ and hence corresponds to the Dyck path

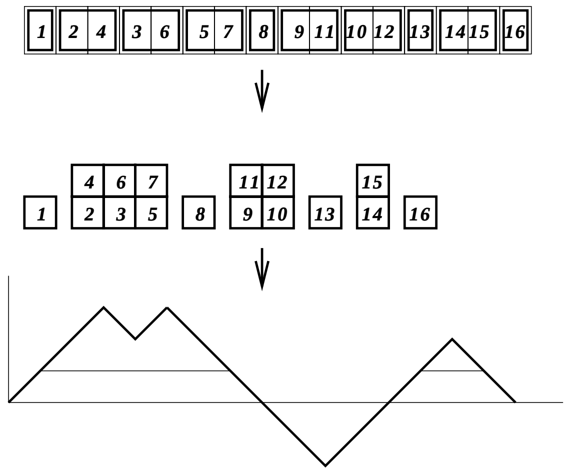

Figure 9: The bijection $\beta_{n,k}$.

P_2. In general, the path Q_{2j-1} that starts at $(f_{j-1}, 0)$ and ends at $(e_j - 1, 0)$ must stay below the x-axis so that Q_{2j-1} is the flip of some Dyck path P_{2j-1}. Similarly, the path Q_{2j} between $(e_j, 1)$ and $(f_j - 1, 1)$ must either be empty or is a path which stays above the line $x = 1$ and hence corresponds to the Dyck path P_{2j}. Finally, the path Q_{2k+1} which follows $(f_k, 0)$ is either empty or is a path that ends at $(2n, 0)$ and stays below the x-axis and, hence, corresponds to the flip of a Dyck path P_{2k+1}. In this way, we can recover the sequence of paths $P_1, \ldots, P_{2k+1}$, which are either empty or Dyck paths, such that

$$P = \overline{P}_1(1,1)P_2(1,-1)\overline{P}_3(1,1)P_4(1,-1)\ldots\overline{P}_{2k-1}(1,1)P_{2k}(1,-1)\overline{P}_{2k+1}.$$

Then for each i, $T_i = \phi^{-1}(P_i)$ is either a standard tableau or the empty tableau. Using these tableaux and being cognizant of the restrictions on the initial segments of elements of $\mathcal{F}_{2n,2k}$ preceding bricks of size one described above, one can easily reconstruct the 2 line intermediate arrays corresponding to $T_1e_1T_2f_2\ldots T_{2k-1}e_{2k}T_{2k}f_{2k}$ T_{2k+1}. For example, this process is pictured on line 2 of Figure 10. Then we only have to rotate all the bricks of size corresponding to a brick of height 2 by 90 degrees to obtain $\beta_{n,k}^{-1}(P)$. This step is pictured on line 3 of 10.

$\square$

As a consequence of Theorem 9, we have the closed expression for $\mathrm{NM}_{\{1323,123\}}(t,x,y)$.

Theorem 10.

$$NM_{\{1323,123\}}(t,x,y) = \left(\frac{1}{U_{\{1323,123\}}(t,y)}\right)^x \quad \textit{where}$$

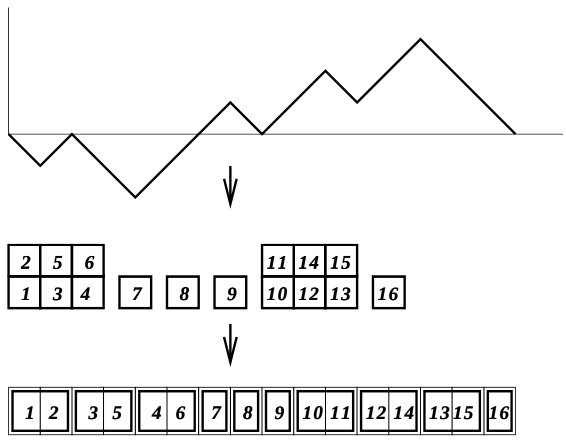

Figure 10: The bijection $\beta_{n,k}^{-1}$.

$$U_{\{1323,123\}}(t,y) = 1 + \sum_{n\geq 1} \frac{t^{2n}}{(2n)!} \left(\sum_{k=0}^{n} \frac{(2k+1)\binom{2n}{n-k}}{n+k+1} (-y)^{n+k} \right) + \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} \left(\sum_{k=0}^{n} \frac{2(k+1)\binom{2n+1}{n-k}}{n+k+2} (-y)^{n+k+1} \right).$$

The proof of Theorem 2.

Let $p \geq 5$ and $\Gamma_p = \{1324\ldots p, 123\ldots p-1\}$. It follows from Lemma 5 that any brick in a fixed point of I_{Γ_p} has size less than or equal to $p-2$.

Let (B,σ) be a fixed point of I_{Γ_p} where $B = (b_1,\ldots,b_t)$ and $\sigma = \sigma_1\cdots\sigma_n$. Suppose that $b_1 = k$ where $1 \leq k \leq p-2$. If $b_1 = 1$, then $\sigma_1 = 1$ and we can remove brick b_1 from (B,σ) and subtract 1 from the remaining elements to obtain a fixed point O' of I_{Γ_p} of length $n-1$. It is easy to see that such fixed points contribute $-yU_{\Gamma_p,n-1}(y)$ to $U_{\Gamma_p,}(y)$.

Next assume that $2 \leq k \leq p-2$. First we claim that $1,\ldots,k-1$ must be in b_1. That is, since the minimal elements in the bricks increase, reading from left to right, and the elements within each brick are increasing, it follows that the first element of brick b_2 is smaller than every element of σ to its right. Thus if there is an increase between bricks b_1 and b_2, it must be the case the elements in brick b_1 are the k smallest elements. If there is a decrease between bricks b_1 and b_2, then there must be a $1324\ldots p$-match that lies in the cells of b_1 and b_2 which must start at position $k-1$. Thus $\sigma_{k-1} < \sigma_{k+1}$ which means that $\sigma_1,\ldots,\sigma_{k-1}$ must be the smallest $k-1$ elements. We then have two cases depending on the position of k in σ.

Case 1. k is in the k^{th} cell of (B,σ).

In this case, if we remove the entire brick b_1 from (B,σ) and subtract k from the numbers in the remaining cells, we will obtain a fixed point O' of $I_{\Gamma_p,n-k}$. It then easily follows that fixed points in Case 1 will contribute $-yU_{\Gamma_p,n-k}(y)$ to $U_{\Gamma_p,n}(y)$.

Case 2. k is in cell $k+1$ of (B,σ).

In this case, it is easy to see that k is in the first cell of the second brick in (B, σ) and there must be a $1324\ldots p$-match between the cells of the first two bricks. This match must start from cell $k-1$ in O with the numbers $k-1$ and k playing the roles of 1 and 2, respectively, in the match. This forces the brick b_2 to have exactly $p-2$ cells. Thus we have two subcases.

Subcase 2.a. There is no $1324\ldots p$-match in (B, σ) starting at cell $k+p-3$

In this case, we claim that $\{\sigma_1, \ldots, \sigma_{k+p-2}\} = \{1, \ldots, k+p-2\}$. That is, we know that the element in the first cell of brick b_3 is smaller than any of the elements of σ to its right. Moreover, if there was a decrease between brick b_2 and b_3, then there must be a $1324\ldots p$-match starting in cell $k+p-3$. Since we are assuming there is not such a match this means that there is an increase between bricks b_2 and b_3. Since the last element of b_2 must be the largest element in either brick b_1 or b_2, it follows that $\{\sigma_1, \ldots, \sigma_{k+p-2}\} = \{1, \ldots, k+p-2\}$. This forces that $\sigma_i = i$ for $i \leq k-1$, $\sigma_k = k+1$, $\sigma_{k+1} = k$, $\sigma_{k+2} = k+2$, $\sigma_i = i$ for $k+2 < i \leq k+p-2$. Hence, the first two bricks of (B, σ) are completely determined. It then follows that if we let O' be the result by removing the first $k+p-2$ cells from (B, σ) and subtracting $k+p-2$ from the numbers in the remaining cells, then O' will be a fixed point in $\mathcal{O}_{\Gamma_p, n-k-(p-2)}$. It then easily follows that fixed points in Subcase 2.a contribute $(-y)^2 U_{\Gamma_p, n-k-(p-2)}(y)$ to $U_{\Gamma_p, n}(y)$.

Subcase 2.b. There is a $1324\ldots p$-match in (B, σ) starting at cell $k+p-3$

In this case, it must be that $\sigma_{k+p-3} < \sigma_{k+p-1} < \sigma_{k+p-2}$ so that there is a decrease between bricks b_2 and b_3. This means that the $1324\ldots p$-match starting in cell $k+p-3$ must be contained in bricks b_2 and b_3. In particular, this means that $b_3 = p-2$. In general, suppose that the bricks $b_2, \ldots, b_{m-1}$ all have exactly $p-2$ cells and let $c_i = k+(i-1)(p-2)-1$ for all $1 \leq i \leq m-1$, so that c_i is the second-to-last cell of brick b_i. In addition, suppose there are $1324\ldots p$-matches starting at cells $c_1, c_2, \ldots, c_{m-1}$ but there are no $1324\ldots p$-match starting at cell $c_m = k-(m-1)(p-2)-1$ in O. We then have the situation pictured in Figure 11 below.

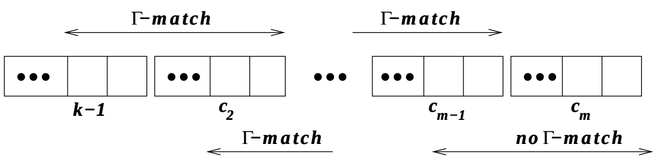

Figure 11: A fixed point with Γ_p-matches starting at c_i for $i = 1, \ldots, m-1$.

First, we claim that $\{\sigma_1, \sigma_2, \ldots, \sigma_{c_{m+1}}\} = \{1, 2, \ldots, c_{m+1}\}$. Since there is no Γ_p-match starting at σ_{c_m} in σ, it cannot be that there is decrease between brick b_m and b_{m+1}. Because the minimal elements in the bricks of B increase, reading from left to right, and the elements in each brick increase, it follows that σ_{c_m+2}, which is the first element of brick b_{m+1}, is smaller than all the elements to its right. On the other hand, because there are $1324\cdots p$-matches starting in σ starting at $c_1, \ldots, c_{m-1}$ it follows that σ_{c_m+1}, which is last cell in brick b_m, is greater than all elements of σ to its left. It follows that $\{\sigma_1, \sigma_2, \ldots, \sigma_{c_{m+1}}\} = \{1, 2, \ldots, c_{m+1}\}$.

Next we claim that we can prove by induction that $\sigma_{c_i} = c_i$ and $\{\sigma_1, \ldots, \sigma_{c_i}\} = \{1, \ldots, c_i\}$ for $1 \leq i \leq m$. Our arguments above show that $\sigma_i = i$ for $i = 1, \ldots, k-1 = c_1$. Thus the base case holds. So assume that $\sigma_{c_{j-1}} = c_{j-1}$, for $1 \leq i \leq j$, and $\{\sigma_1, \sigma_2, \ldots, s_{c_{j-1}}\} = \{1, 2, \ldots, c_{j-1}\}$. Since there is a $132 \cdots p$-match in σ starting at position c_{j-1} and $p \geq 5$, it must be the case that all the numbers $\sigma_{c_{j-1}}, \sigma_{c_{j-1}+1}, \ldots, \sigma_{c_{j-1}+p-3}$ are all less than $\sigma_{c_j} = \sigma_{c_{j-1}+p-2}$. Since $\{\sigma_1, \sigma_2, \ldots, \sigma_{c_{j-1}}\} = \{1, 2, \ldots, c_{j-1}\}$, we must have $\sigma_{c_j} \geq c_j$. If $\sigma_{c_j} > c_j$, then let d be the smallest number from $\{1, 2, \ldots, c_j\}$ that does not belong to the bricks $b_1, \ldots, b_j$. Since the numbers in a brick increase and the first cells of the bricks form an increasing sequence, it must be the case that d is in the first cell of brick b_{j+1}, namely $\sigma_{c_j+2} = d$. We have two possibilities for j.

1. If $j < m$, then $\sigma_{c_j+2} = d < c_j \leq \sigma_{c_j}$. This contradicts the assumption that there is a $1324 \ldots p$-match starting from cell c_j in σ for σ_{c_j} needs to play the role of 1 in such a match.

2. If $j = m$, then there is a descent between the bricks b_m and b_{m+1} and there must be a $1324 \ldots p$-match that lies entirely in the cells of b_m and b_{m+1} in O. However, the only possible match must start from cell c_m, the second-to-last cell in b_m. This contradicts our assumption that there is no match starting from cell c_m in O.

Hence, $\sigma_{c_j} = c_j$ and $\{\sigma_1, \sigma_2, \ldots, \sigma_{c_j}\} = \{1, 2, \ldots, c_j\}$. for $1 \leq j \leq m$.

We claim that the values of σ_i are forced for $i \leq c_m + 1$. That is, consider the first $1324 \cdots p$-match starting at position $k-1$. Since $p \geq 5$, we know that $\sigma_{k+p-2} = k + p - 2 > \sigma_{k+2}$. This forces that $\sigma_k = k+1$, $\sigma_{k+1} = k$, $\sigma_{k+2} = k+2$ so that the values of σ_i for $i \leq k+p-2$. This type of argument can be repeated for all the remaining $1324 \cdots p$-matches starting at $c_2, \ldots, c_{m-1}$. Thus if we remove the first $k + (m-1)(p-2)$ cells of O, we obtain a fixed point O' of I_{Γ_p} in $\mathcal{O}_{\Gamma_p, n-k-(m-1)(p-2)}$. On the other hand, suppose that we start with a fixed point (D, τ) of I_{Γ_p} in $\mathcal{O}_{\Gamma_p, n-k-(m-1)(p-2)}$ where $D = (d_1, \ldots, d_r)$ and $\tau = \tau_1, \ldots, \tau_{n-k-(m-1)(p-2)}$. Let $\overline{\tau} = \overline{\tau}_1 \cdots \overline{\tau}_{n-k-(m-1)(p-2)}$ be the result of adding $n-k-(m-1)(p-2)$ to every element of τ. Then it is easy to see that (B, σ) is a fixed point of I_{Γ_p}, where $B = (k, (p-2)^m, d_1, \ldots, d_r)$ and $\sigma = \sigma_1 \cdots \sigma_{k+(m-1)p-2}\overline{\tau}$ where $\sigma_1 \cdots \sigma_{k+(m-1)(p-2)}$ is the unique permutation in $\mathfrak{S}_{k+(m-1)(p-2)}$ with $1324 \cdots p$-matches starting at positions $c_1, \ldots, c_{m-1}$. It follows that the contribution of the fixed points in Case 2.b to $U_{\Gamma_p,n}(y)$ is $\sum_{m \geq 3} (-y)^m U_{\Gamma_p, n-k-(m-1)(p-2)(y)}$.

Hence, for any fixed point O_k that has k cells in the first brick, for $1 \leq k \leq p-2$, the contribution of O_k to $U_{\Gamma_p,n}(y)$ is

$$(-y)U_{\Gamma_p,n-k}(y) + \sum_{m=2}^{\lfloor \frac{n-k}{p-2} \rfloor} (-y)^m U_{\Gamma_p, n-k-(m-1)(p-2)}(y).$$

Therefore, we obtain the following recursion for $U_{\Gamma_p,n}(y)$

$$U_{\Gamma_p,n}(y) = \sum_{k=1}^{p-2}(-y)U_{\Gamma_p,n-k}(y) + \sum_{k=1}^{p-2}\sum_{m=2}^{\lfloor\frac{n-k}{p-2}\rfloor}(-y)^m U_{\Gamma_p,n-k-(m-1)(p-2)}(y).$$

This completes the proof of Theorem 2. $\square$

6 Conclusion and Problems for Future Research

In this paper, we have shown that the reciprocal method introduced by Jones and Remmel in [11] can be extended to a family Γ whose permutations all start with 1 and have at most one descent. Specifically, we have proved if $\Gamma = \Gamma_{k_1,k_2} =$

$$\{\sigma \in \mathfrak{S}_p : \sigma_1 = 1, \sigma_{k_1+1} = 2, \sigma_1 < \sigma_2 < \cdots < \sigma_{k_1} \ \&\ \sigma_{k_1+1} < \sigma_{k_1+2} < \cdots < \sigma_p\}$$

where $k_1, k_2 \geq 2$, $\Gamma = \Gamma_{k_1,k_1,s} = \Gamma_{k_1,k_2} \cup \{1\cdots s(s+1)\}$ where $s \geq k_1 \geq 2$, or $\Gamma = \Gamma_p = \{1324\cdots p, 123\cdots p-1\}$ where $p \geq 4$, then the polynomials $U_{\Gamma,n}(y)$ satisfy simple recursions and these recursions can be used to compute the terms in the generating function

$$\mathrm{NM}_\Gamma(t,x,y) = \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\Gamma)} x^{\mathrm{LRmin}(\sigma)}y^{1+\mathrm{des}(\sigma)}.$$

From the values of the polynomials $U_{\Gamma,n}(y)$ computed through Mathematica, we conjecture that the polynomials $U_{\Gamma,n}(y)$ are log-concave for $\Gamma = \{1324, 1423\}$ and $\Gamma = \{1324\cdots p, 123\cdots p\}$, where $p \geq 4$. However, the polynomials $U_{\Gamma_{k_1,k_2},n}(-y)$ are not always log-concave when k_1 is larger than k_2.

The next set of problems to consider is to show that the same machinery can be extended to families Γ of permutations which all start with 1 but may have more than one descent. This type of problem in the case where Γ consists of single permutation τ was first mentioned by Jones and Remmel in [14], where the authors gave a recursion for the polynomial $U_{\tau,n}(y)$ for $\tau = 15243$.

The main problem when the permutations in a family Γ are allowed to have more than one descent is that the mapping I_Γ defined in Section 3 is no longer an involution. To see this, suppose the permutations in Γ have more than one descents and consider the case where we have a decrease between the last cell of brick b_{i-1} and the first cell of brick b_i, but we are unable to combine them since there is a Γ-match that involves the cells of bricks b_{i-1} and b_i. In this case, brick b_i will have at least one cell labeled with y. According to the current mapping, we will try to split brick b_i after some cell c labeled with y into two bricks: b', containing all the cells of b_i up to and including c, and b'', containing all the remaining cells of b_i. Then, we will be able to combine b' with b_{i-1} because there is still a decrease between b_{i-1} and b' but now there is no Γ-match that lies in the cells of b_{i-1} and b'. This means that we cannot use cell c in a definition of an involution. Thus we must restrict ourselves to cells c labeled with y which do not have this property. The result of this restriction is that the fixed points are more complicated than before. In particular, we can no longer guarantee that if (B, σ) is a fixed point of such an involution, then σ is increasing in the bricks of B. Nevertheless one can analyze the fixed points of such an involution for certain simple permutations τ and simple families of permutations Γ. For example, we can prove the following results.

Theorem 11. *For $\tau = 1432$, $U_{\tau,1}(y) = -y$, and for $n \geq 2$,*

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) - y^2\binom{n-2}{2}U_{\tau,n-3}(y).$$

Theorem 12. *For $\tau = 142536$, $U_{\tau,1}(y) = -y$, and for $n \geq 2$,*

$$\begin{aligned} U_{\tau,n}(y) = \quad & (1-y)U_{\Gamma,n-1}(y) + \sum_{k=1}^{\lfloor (n-2)/6 \rfloor} H_{2k} y^{3k} U_{n-6k-1}(y) \\ & - \sum_{k=1}^{\lfloor n/6 \rfloor} H_{2k-1} y^{3k-1} \left[U_{\tau,n-6k+2}(y) + yU_{\tau,n-6k+1}(y)\right] \end{aligned}$$

where H_i is the determinant the matrix of Catalan numbers, given by the following formulas.

$$H_{2k-1} = \begin{vmatrix} C_2 & C_5 & C_8 & C_{11} & \cdots & C_{3k-4} & C_{3k-2} \\ -1 & C_2 & C_4 & C_8 & \cdots & C_{3k-7} & C_{3k-5} \\ 0 & -1 & C_2 & C_5 & \cdots & C_{3k-10} & C_{3k-8} \\ 0 & 0 & -1 & C_2 & \cdots & C_{3k-13} & C_{3k-11} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_2 & C_4 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{vmatrix}, \text{ and}$$

$$H_{2k} = \begin{vmatrix} C_2 & C_5 & C_8 & C_{11} & \cdots & C_{3k-4} & C_{3k-1} \\ -1 & C_2 & C_5 & C_8 & \cdots & C_{3k-7} & C_{3k-4} \\ 0 & -1 & C_2 & C_5 & \cdots & C_{3k-10} & C_{3k-7} \\ 0 & 0 & -1 & C_2 & \cdots & C_{3k-13} & C_{3k-10} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_2 & C_5 \\ 0 & 0 & 0 & 0 & \cdots & -1 & C_2 \end{vmatrix}.$$

Theorem 13. *For $\tau = 162534$, $U_{\tau,1}(y) = -y$, and for $n \geq 2$,*

$$\begin{aligned} U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) - \sum_{k=1}^{\lfloor n/6 \rfloor} y^{3k-1}\binom{n-3k-1}{3k-1} U_{\tau,n-6k+1}(y) \\ + \sum_{k=1}^{\lfloor (n-2)/6 \rfloor} y^{3k}\binom{n-3k-2}{3k} U_{\tau,n-6k-1}(y). \end{aligned}$$

Theorem 14. *For* $\Gamma = \{14253, 15243\}$, $U_{\Gamma,1}(y) = -y$, *and for* $n \geq 2$,

$$U_{\Gamma,n}(y) = (1-y)U_{\Gamma,n-1}(y) - y^2(n-3)\left(U_{\Gamma,n-4}(y) + (1-y)(n-5)U_{\Gamma,n-5}(y)\right) - y^3(n-3)(n-5)(n-6)U_{\Gamma,n-6}(y).$$

These results will appear in subsequent papers.

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