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On the punctured power graph of a finite group

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Abstract

The punctured power graph $\mathcal{P}^\star(G)$ of a finite group G is the graph which has as vertex set the nonidentity elements of G, where two distinct elements are adjacent if one is a power of the other. We show that $\mathcal{P}^\star(G)$ has diameter at most 2 if and only if G is nilpotent and every Sylow subgroup of G is either a cyclic group or a generalized quaternion 2-group. Also, we show that if G is a finite group and $\mathcal{P}^\star(G)$ has diameter 3, then G is not simple. Finally, we show that $\mathcal{P}^\star(G)$ is Eulerian if and only if G is a cyclic 2-group or a generalized quaternion 2-group.

1 Introduction

In this paper we present particular properties of punctured power graphs. Throughout, G shall denote a finite group, and e shall denote its identity.

Definition 1 The *directed power graph* $\overrightarrow{\mathcal{P}}(G)$ of G is the directed graph whose vertex set is the underlying set of G with an edge (u, v) for all distinct u, $v \in G$ where v is a power of u. The (undirected) *power graph* $\mathcal{P}(G)$ of G has the same vertex set with an (undirected) edge between distinct group elements when one is a power of the other in G. The *punctured power graph* $\mathcal{P}^\star(G)$ of G is the subgraph of $\mathcal{P}(G)$ induced on $G \setminus \{e\}$.

Directed and undirected power graphs of semigroups were introduced in [7], and some combinatorial properties were examined in [8, 9, 10]. Undirected power graphs of groups were studied in [2, 3, 4, 5, 11]. The reference [1] surveys the literature to date. Standard references provide the necessary background, such as [13] for graphs and [12] for groups.

2 Shortest paths

We consider properties of the shortest paths in punctured power graphs.

Definition 2 Pick adjacent vertices a, $b \in \mathcal{P}(G)$. Write $a \to b$ or $b \leftarrow a$ if b is a power of a and $a \leftrightarrow b$ when each is a power of the other. Write $a \rightarrowtail b$ or $b \leftarrowtail a$ when $a \to b$ but $b \not\to a$. We augment a path $x_0 x_1 \cdots x_d$ by adding appropriate arrows between vertices.

Let $\langle x \rangle$ denote the cyclic subgroup of G generated by $x \in G$, and let $\langle\langle x \rangle\rangle$ denote the set of generators of $\langle x \rangle$. Note that $x \to y$ if and only if $\langle y \rangle \subseteq \langle x \rangle$; $x \rightarrowtail y$ if and only if $\langle y \rangle \subsetneq \langle x \rangle$; and $x \leftrightarrow y$ if and only if $\langle x \rangle = \langle y \rangle$ if and only if $y \in \langle\langle x \rangle\rangle$. Let $\mathrm{o}(x)$ to denote the order of x as an element in G, and let ϕ denote the Euler totient function. Then $|\langle\langle x \rangle\rangle| = \phi(\mathrm{o}(x))$.

Lemma 3 *Suppose $x = x_0 x_1 \cdots x_d = y$ is a shortest path between x and y in $\mathcal{P}^\star(G)$. Then for all i $(1 \le i \le d-1)$, either $x_{i-1} \rightarrowtail x_i \leftarrowtail x_{i+1}$ or $x_{i-1} \leftarrowtail x_i \rightarrowtail x_{i+1}$.*

Proof. If $x_{i-1} \to x_i \to x_{i+1}$, then $x_{i+1} = x_i^n = (x_{i-1}^m)^n$ for some positive integers m and n, so $x_{i-1} \to x_{i+1}$. This implies that omitting x_i gives a shorter path from x to y, contradicting the minimality of the length of the given path. The oppositely directed case is similar. $\square$

Corollary 4 *With reference to Lemma 3, if one of x_{i-1} or x_{i+1} has prime order for some i $(1 \le i \le d-1)$, then $x_{i-1} \leftarrowtail x_i \rightarrowtail x_{i+1}$.*

Proof. Suppose $x_{i-1} \to x_i$ and x_{i-1} has prime order. Then x_i generates the same subgroup, so $x_{i-1} \leftrightarrow x_i$, contradicting Lemma 3. The case where x_{i+1} has prime order is similar. $\square$

Lemma 5 *If $x = x_0 \cdots x_{i-1} x_i x_{i+1} \cdots x_d = y$ is a shortest path between x and y in $\mathcal{P}^\star(G)$, then for all i $(1 \le i \le d-1)$ and for all $x_i' \in \langle\langle x_i \rangle\rangle$, $x = x_0 \cdots x_{i-1} x_i' x_{i+1} \cdots x_d$ $= y$ is too.*

Proof. Pick $x_i' \in \langle\langle x_i \rangle\rangle$, and say $x_i' = x_i^k$ and $x_i = {x_i'}^\ell$. Suppose $x_{i-1} \rightarrowtail x_i \leftarrowtail x_{i+1}$, with $x_i = x_{i-1}^m$ and $x_i = x_{i+1}^n$. Then $x_i' = x_{i-1}^{km}$ and $x_i' = x_{i+1}^{kn}$, so $x_{i-1} \rightarrowtail x_i' \leftarrowtail x_{i+1}$. The case $x_{i-1} \leftarrowtail x_i \rightarrowtail x_{i+1}$ is treated similarly. $\square$

Lemma 6 *Suppose that $\mathcal{P}^\star(G)$ is connected. The shortest path distance in $\mathcal{P}^\star(G)$ between distinct nonadjacent vertices of prime order is even.*

Proof. Let x and y be distinct nonadjacent elements of prime order. The arrows of any augmented shortest path between them point to x and y by Corollary 4. By Lemma 3 the arrows alternately point to and away from successive vertices. This requires that their distance is even. □

Lemma 7 *Adjacent vertices in $\mathcal{P}(G)$ commute in G. If $x \leftarrow y \rightarrow z$, the x, y, z commute.*

Proof. If x and y are adjacent, then one is a power of the other, and if $x \leftarrow y \rightarrow z$ then all three are powers of y. □

3 Punctured power graphs of low diameter

Recall that the *diameter* of a graph is the maximum distance between any pairs of vertices. A graph has diameter 1 precisely when it is complete. We say that a disconnected graph has diameter ∞, so any graph of finite diameter is necessarily connected. The punctured power graph of any dihedral group is disconnected since the flips are isolated vertices. We focus on groups with low diameter punctured power graphs. The diameter of $\mathcal{P}^\star(G)$ is at least as large as that of $\mathcal{P}(G)$.

Theorem 8 *[4, Theorem 2.12] $\mathcal{P}(G)$ has diameter at most 2. Moreover, $\mathcal{P}(G)$ has diameter 1 if and only if G is a cyclic p-group.*

Corollary 9 *$\mathcal{P}^\star(G)$ has diameter 1 if and only if G is a cyclic p-group.*

Proof. In $\mathcal{P}(G)$, every nonidentity element of G is adjacent to the identity. Thus $\mathcal{P}(G)$ is complete if and only if $\mathcal{P}^\star(G)$ is complete. □

We recall a connection between a family of groups and a graph theoretic property. A group is called a *generalized quaternion group* if it has a presentation $\langle x, y \mid x^{2n} = y^4 = 1, x^n = y^2, y^{-1}xy = x^{-1}\rangle$ for some integer $n \geq 2$. Such a group is a 2-group when n is a power of 2.

Theorem 10 *A p-group has a unique subgroup of order p if and only if it is a cyclic group or a generalized quaternion 2-group*

A graph is 2-*connected* if the removal of any one vertex does not make the graph disconnected. By construction $\mathcal{P}(G)$ is 2-connected if and only if $\mathcal{P}^\star(G)$ is connected.

Theorem 11 *[11, Theorem 7] The power graph of a p-group is 2-connected if and only if the p-group is a cyclic group or a generalized quaternion 2-group.*

Lemma 12 *Suppose that $\mathcal{P}^\star(G)$ has diameter at most 3. Then the Sylow subgroups of G are cyclic groups or generalized quaternion 2-groups.*

Proof. If $\mathcal{P}^\star(G)$ has diameter 1, then the result follows from Corollary 9. Assume $\mathcal{P}^\star(G)$ has diameter at least 2. Let p be a prime divisor of $|G|$, and let Q be a Sylow p-subgroup of G. By Theorem 11 and the comment just prior to it, it suffices to show that $\mathcal{P}^\star(Q)$ is connected.

Pick nonidentity nonadjacent elements x, $y \in Q$. Let x' be x if $\mathrm{o}(x) = p$ and $x^{\mathrm{o}(x)/p}$ otherwise, and let y' be y if $\mathrm{o}(y) = p$ and $y^{\mathrm{o}(y)/p}$ otherwise. Then x' and y' are elements of Q with order p. Assume $x' \neq x$, $y' \neq y$; the remaining cases are similar. If $x' = y'$, then $xx'y$ is a path in $\mathcal{P}^\star(Q)$. If x' and y' are distinct but adjacent in $\mathcal{P}^\star(G)$, then it must be the case that y' is a power of x', so $xx'y$ is a path in $\mathcal{P}^\star(Q)$. Suppose x' and y' are distinct and nonadjacent but have a common neighbor z in $\mathcal{P}^\star(G)$. By Corollary 4, $x' \leftarrowtail z \rightarrowtail y'$, so $\langle x'\rangle$ and $\langle y'\rangle$ are cyclic subgroups $\langle z\rangle$ of the same order. Thus $\langle x'\rangle = \langle y'\rangle$, so $x' \leftrightarrow y'$. Thus $xx'y'y$ is a path in $\mathcal{P}^\star(Q)$. Note that x', y' are at an even distance at most three by assumption and by Lemma 6. Thus we have exhausted all possibilities and have shown that $\mathcal{P}^\star(Q)$ is connected, as required. □

Theorem 13 *A finite group is nilpotent if and only if it is the direct product of its Sylow subgroups, if and only if every pair of elements with coprime order commute. In particular, abelian groups and p-groups are nilpotent.*

Theorem 14 *The following are equivalent.*

- **(i)** *$\mathcal{P}^\star(G)$ has diameter at most 2.*
- **(ii)** *G is nilpotent and all of its Sylow subgroups are cyclic groups or generalized quaternion 2-groups.*

When (i) and (ii) hold, $\mathcal{P}(G)$ and $\mathcal{P}^\star(G)$ have the same diameter.

Proof. Note that both (i) and (ii) hold when $\mathcal{P}^\star(G)$ has diameter 1 by Theorem 8 and Corollary 9, so there is nothing to show in this case.

(i)⇒(ii): Suppose $\mathcal{P}^\star(G)$ has diameter 2. We first show that G is nilpotent. Pick x, $y \in \mathcal{P}^\star(G)$ with coprime orders. Note that neither is a power of the other, so they are nonadjacent in $\mathcal{P}(G)$. However, they have a common neighbor z in $\mathcal{P}^\star(G)$ since its diameter is 2. Since xzy is a shortest path, we may appeal to Lemma 3. Now z cannot be a power of both x and y since in this case its order would divide both those of x and y. Thus x and y are both powers of z, so they commute. Hence G is nilpotent by Theorem 13. Lemma 12 gives that its Sylow subgroups are cyclic groups or generalized quaternion 2-groups.

(ii)⇒(i): Suppose that G is nilpotent and that its Sylow subgroups are cyclic groups or generalized quaternion 2-groups. Pick x, $y \in \mathcal{P}^\star(G)$. If $(\mathrm{o}(x), \mathrm{o}(y)) = 1$, then $xy \rightarrowtail x$ and $xy \rightarrowtail y$ by Theorem 13, so their distance is at most 2. If some prime p divides $(\mathrm{o}(x), \mathrm{o}(y))$, then there exist natural numbers r, s such that $x' = x^r$ and $y' = y^s$ have order p since G is nilpotent. By Theorem 10, x' and y' are generators the unique subgroup of order p, so $x' \leftrightarrow y'$ and $xx'y$ is a path of length 2 in $\mathcal{P}^\star(G)$. □

Corollary 15 *If $\mathcal{P}^\star(G)$ has diameter 3, then G is not nilpotent.*

Proof. This is clear from Lemma 12 and Theorem 14. □

Lemma 16 *If $\mathcal{P}^\star(G)$ has diameter at most 3, then elements of G with prime order commute.*

Proof. Pick x, $y \in G$ with prime order. Their distance in $\mathcal{P}^\star(G)$ is at most 2 by Lemma 6. There is nothing to show if $x = y$. If x and y are adjacent, then one is a power of the other, so they commute. If x and y are nonadjacent, then there is some nonidentity $z \in G$ adjacent to both. By Corollary 4, $x \longleftarrow\!\!\prec z \succ\!\!\longrightarrow y$, so x and y commute by Lemma 7. □

Corollary 17 *If $\mathcal{P}^\star(G)$ has diameter at most 3, then for any square-free divisor f of $|G|$, the product of group elements of the appropriate distinct prime orders has order f.*

4 Comments on non-simplicity

We give a couple of power graph criteria for the non-simplicity of G.

Theorem 18 *[6, Theorem B] Let G be a finite non-abelian simple group. Then there exist distinct prime divisors a, b of $|G|$ such that for all $x, y \in G$ with $\mathrm{o}(x) = a$ and $\mathrm{o}(y) = b$, the subgroup $\langle x, y\rangle$ is nonsolvable.*

Corollary 19 *If G is a non-abelian simple group, then $\mathcal{P}^\star(G)$ has diameter at least 4.*

Proof. Suppose $\mathcal{P}^\star(G)$ has diameter at most 3. For all distinct prime divisors a, b of $|G|$, the elements of order a commute with those of order b by Lemma 16. In particular, the subgroup generated by such elements is abelian, and so solvable. The result follows by Theorem 18. □

Lemma 20 *Let p be the greatest prime divisor of $|G|$. If some vertex has at least $|G|/p$ neighbors in $\mathcal{P}(G)$, then G is not a non-abelian simple group.*

Proof. Suppose x has at least $|G|/p$ neighbors. By Lemma 7, x and its neighbors are in $C_G(x)$, so $|C_G(x)| > |G|/p$ and $r = |G : C_G(x)| < p$. If G is a finite non-abelian simple group, then $r \geq 3$ and G is isomorphic to a subgroup of the alternating group $\mathbb{A}_r$. This cannot be since p does not divide $r!$, so the result holds. □

Corollary 21 *Let q be the least prime divisor of $|G|$. Any vertex with at least $|G|/q$ neighbors in $\mathcal{P}(G)$ is in the center of G.*

Proof. If x has at least $|G|/q$ neighbors, then $|G : C_G(x)| < q$ by Lemma 7, so $C_G(x) = G$. □

5 Eulerian punctured power graphs

A graph is said to be *Eulerian* whenever there is a closed walk that traverses every edge of the graph exactly once. A connected graph is Eulerian if and only if every vertex has even degree. It is easy to see that $\mathcal{P}(G)$ is Eulerian if and only if $|G|$ is odd [11, Lemma 1]. We determine when $\mathcal{P}^\star(G)$ is Eulerian. Recall that the *degree* of a vertex in a graph is the number of adjacent vertices.

Lemma 22 *Pick $x \in \mathcal{P}^\star(G)$, and let $\deg^\star(x)$ denote its degree in $\mathcal{P}^\star(G)$. Then*

$$\deg^\star(x) = \mathrm{o}(x) - 2 + \sum_{g \in G,\, x \leftarrow\!\!\prec g} \varphi(\mathrm{o}(g)). \tag{1}$$

Proof. Observe $\deg^\star(x) = |\{g \in G \,|\, x \to g\}| - |\{e\}| + |\{g \in G \,|\, x \leftarrow g\}| - |\{g \in G \,|\, x \leftrightarrow g\}|$ (last term eliminates double counts). By Definition 2 and the comments following it, $|\{g \in G \,|\, x \to g\}| = \mathrm{o}(x) - 1$ and $|\{g \in G \,|\, x \leftarrow g\}| - |\{g \in G \,|\, x \leftrightarrow g\}| = \sum_{g \in G, x \leftarrow\!\!\prec g} \varphi(\mathrm{o}(g))$. □

Corollary 23 *For all $x \in \mathcal{P}^\star(G)$, $\mathrm{o}(x)$ and $\deg^\star(x)$ have the same parity.*

Proof. For all $g \in G$ with $x \leftarrow\!\!\prec g$, $\phi(\mathrm{o}(g))$ is even since $\mathrm{o}(g) > \mathrm{o}(x) \geq 2$. Thus the sum on the right side of (1) is even, so the result follows. □

Theorem 24 *$\mathcal{P}^\star(G)$ is Eulerian if and only if G is a cyclic 2-group or a generalized quaternion 2-group.*

Proof. Suppose $\mathcal{P}^\star(G)$ is Eulerian. Then every element of $\mathcal{P}^\star(G)$ has even order by Corollary 23. Hence G is a 2-group. Moreover, Eulerian implies connected, so G is a cyclic 2-group or a generalized quaternion 2-group by Theorem 11.

Suppose G is a cyclic 2-group or a generalized quaternion 2-group. Every non-identity element has even order, and hence even degree in $\mathcal{P}^*(G)$ by Corollary 23. By Theorem 10, there is a unique element of order 2, and it is adjacent to every vertex of $\mathcal{P}^\star(G)$. Thus $\mathcal{P}^\star(G)$ is connected, and hence Eulerian. □

This paper should be seen as part of a program to understand the connections between a group and its power graph. Ultimately, we hope to find combinatorial conditions on a power graph which correspond to interesting group properties. This paper, like many others (see the survey [1]), starts from strong graph theoretic conditions on a power graph (low diameter, Eulerian, etc.) and finds that only a few special groups give rise to such a power graph. Our hope is to gain insight so that we might eventually tackle more interesting problems, such as finding a criterion or characterization for solvable or simple groups from combinatorial properties of their power graphs.

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