Symmetric alternating sign matrices

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Abstract

In this note we consider completions of $n \times n$ symmetric (0, -1)-matrices to symmetric alternating sign matrices by replacing certain 0s with +1s. In particular, we prove that any $n \times n$ symmetric (0, -1)-matrix that can be completed to an alternating sign matrix by replacing some 0s with +1s can be completed to a symmetric alternating sign matrix. Similarly, any $n \times n$ symmetric (0, +1)-matrix that can be completed to an alternating sign matrix by replacing some 0s with -1s can be completed to a symmetric alternating sign matrix.

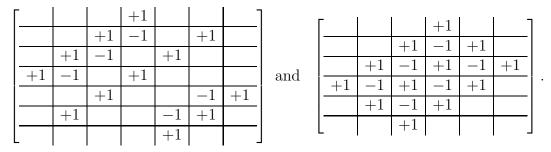
1 Introduction

An alternating sign matrix, abbreviated ASM, is an $n \times n$ (0, +1, -1)-matrix such that, ignoring 0s, in each row and column, the +1s and -1s alternate, beginning and ending with a +1. An ASM cannot contain any -1s in rows 1 and n and columns 1 and n. The book [1] by Bressoud contains a history of the development of ASMs.

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In [2], there is an investigation of the zero-nonzero patterns of ASMs. The paper [3] considers the problem of completing a (0, -1)-matrix to an ASM by replacing some 0s with +1s. Each row and column of an ASM contains an odd number of nonzeros with the first and last rows and columns each containing exactly one nonzero and that nonzero is a +1. If an ASM (regarded as a square) is subjected to any of the symmetries of a square (the dihedral group), the result is also an ASM.

The simplest examples of ASMs are the permutation matrices. Other examples of ASMs are



(For visual clarity, we usually block off rows and columns and then suppress the 0s in (0, +1, -1)-matrices.)

Our emphasis in this note is on combinatorial properties of symmetric ASMs, of which the preceding two ASMs are examples. Given an $n \times n$ (0, -1)-matrix A, any matrix B obtained from A by replacing some 0s by +1s is a (+1)-completion of A; if B is an ASM, then B is called a (+1)-completion of A to an ASM or an ASM (+1)completion of A. In [3] ASM (+1)-completions of (0, -1)-matrices (called, simply, ASM completions) were investigated with an emphasis on the so-called borderedpermutation (0, -1)-matrices. By an $n \times n$ bordered-permutation (0, -1)-matrix Awe mean an $n \times n$ (0, -1)-matrix such that the first and last rows and columns contain only zeros, and the submatrix $A[\{2, 3, \ldots, n-1\}|\{2, 3, \ldots, n-1\}]$ obtained by deleting rows and columns 1 and n is -P where P is a permutation matrix. Here we consider (+1)-completions of symmetric (0, -1)-matrices to symmetric ASMs.

We also consider here completions of an $n \times n$ (0, +1)-matrix A to ASMs by replacing some 0s with -1s. We call these ASM (-1)-completions. In order that A has an ASM (-1)-completion, it is necessary that there be at least one +1 in each row and column, only one +1 in the first and last rows and columns, and no consecutive +1s in a row or column. **Example 1** Let A be the symmetric bordered-permutation (0, -1)-matrix:

Γ	-							
							-1	
						-1		
				-1				
					-1			
			-1					
		-1						
	-							

Then it is straightforward to check that A has a unique (+1)-completion to an ASM and this (+1)-completion is symmetric:

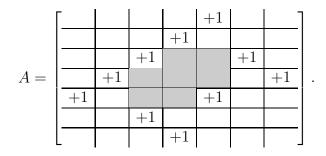
Γ						+1	-
					+1	-1	+1
			+1		-1	+1	
		+1	-1	+1			
			+1	-1	+1		
	+1	-1		+1			
+1	-1	+1					
	+1						

On the other hand, the symmetric (0, -1)-matrix

Γ						-
					-1	
			-1	-1		
		-1				
		-1				
	-1					
						_

does not have a (+1)-completion to an ASM; it suffices to examine rows 1, 2, and 3.

Example 2 Consider the 7×7 symmetric (0, +1)-matrix



The -1s in any (-1)-completion of A to an ASM must be in the shaded positions. Any (-1)-completion of A must have three -1s. There are three (-1)-completions of A, namely, as given below, the matrix A' and its transpose, and the symmetric matrix A'':

	-				+1		-		Γ				+1]
				+1								+1				
			+1	-1		+1					+1		-1	+1		
A' =		+1			-1		+1	, A'' =		+1		-1			+1	.
	+1		-1		+1				+1		-1		+1			
			+1								+1					
	_			+1								+1				

In [3] it was shown that every bordered-permutation (0, -1)-matrix can be (+1)completed to an ASM. We first show that every $n \times n$ symmetric bordered-permutation (0, -1)-matrix can be (+1)-completed to a symmetric ASM and obtain a bound
on the number of such (+1)-completions. There is not an analogue of this result for (-1)-completions, since a permutation matrix is already an ASM. Our main results
are that (i) if a symmetric (0, -1)-matrix has an ASM (+1)-completion, then it also
has a symmetric ASM (+1)-completion, and (ii) if a symmetric (0, +1)-matrix has
an ASM (-1)-completion, then it also has a symmetric ASM (-1)-completion.

2 Symmetric ASM Completions

Theorem 3 Let $A = [a_{ij}]$ be an $n \times n$ symmetric bordered-permutation (0, -1)-matrix. Then A has a (+1)-completion to a symmetric ASM.

Proof. This theorem will follow from Theorem 7 and the theorem in [3] that every bordered-permutation (0, -1)-matrix can be (+1)-completed to an ASM. We give a short independent proof.

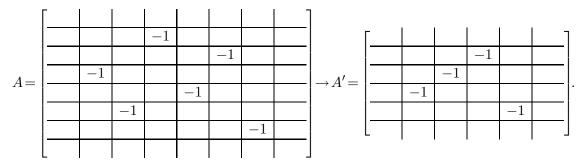
We use induction on n. The theorem is trivial if n = 2 or 3. Let $n \ge 4$. Let k be such that $a_{2k} = a_{k2} = -1$. Let A' = A(2, k|2, k) be the symmetric matrix obtained from A by deleting rows and columns 2 and k. (This matrix is $(n - 1) \times (n - 1)$ if k = 2 and $(n-2) \times (n-2)$ otherwise.) We use for the indices of the row and columns of A' the same indices they had in A; thus the index set for rows and columns of A' is $\{1, 2, \ldots, n\} \setminus \{2, k\}$. By induction A' has a (+1)-completion $B' = [b'_{ij}]$ to a symmetric ASM. Let r be such that $b'_{1r} = b'_{r1} = +1$.

If r > k, let s be the first integer such that $b'_{s,k-1} = b'_{k-1,s} = +1$. We then let B be the matrix which has +1s in all other positions that B' has +1s except for the positions (1, r), (r, 1), (s, k-1), and (k-1, s) and, in addition, has +1s in positions (1, k), (k, 1), (2, k-1), (k-1, 2), (2, r), (r, 2), (s, k), (k, s). Then B is a symmetric ASM (+1)-completion of A.

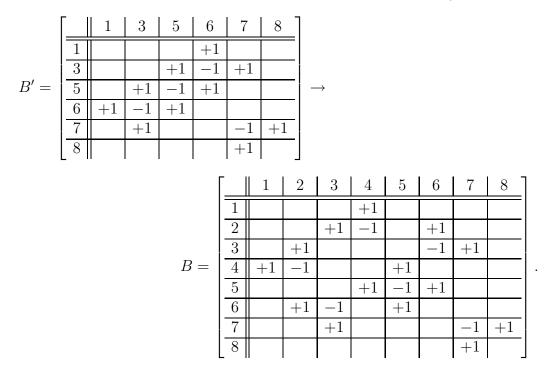
If r < k, let s be the first integer such that $b'_{s,k+1} = b'_{k+1,s} = +1$. We then let B be the matrix which has +1s in all other positions that B' has +1s except for the positions (1, r), (r, 1), (s, k+1), and (k+1, s) and, in addition, has +1s in positions (1, k), (k, 1), (2, k+1), (k+1, 2), (2, r), (r, 2), (s, k), (k, s). Then B is a symmetric ASM (+1)-completion of A.

We give an example illustrating the inductive proof of Theorem 3.

Example 4 In this example, n = 8 and with the above notation, k = 4, r = 6, and s = 5. Let



Then, where we have included the row and column indices for clarity, we have



If A is a symmetric (0, -1)-matrix, then $\pi_s(A)$ denotes the number of (+1)completions of A to a symmetric ASM. By Theorem 3, if A is also a borderedpermutation (0, -1)-matrix, then $\pi_s(A) \ge 1$. We now give an upper bound for $\pi_s(A)$ in general.

Theorem 5 Let $A = [a_{ij}]$ be an $n \times n$ bordered symmetric (0, -1)-matrix such that A has a -1s on the main diagonal and 2b -1s off the main diagonal. Then

$$\pi_s(A) \le \frac{1}{2^{a+b}} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k (n-2k)!k!}.$$
(1)

(The number $\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k (n-2k)!k!}$ is the number of $n \times n$ symmetric permutation matrices [5, p. 218].)

Proof. Let k be the maximum index of a row of A with a -1, and let l be the maximum index of a column with a -1 in row k. Thus $a_{kl} = a_{lk} = -1$ and we let $B = [b_{ij}]$ be the symmetric matrix obtained from A by replacing a_{kl} and a_{lk} with 0s. We show that $\pi_s(A) \leq \frac{\pi_s(B)}{2}$ by establishing, when $\pi_s(A) \neq 0$, a one-to-two correspondence from the set $C_s(A)$ of symmetric ASM (+1)-completions of A to the set $C_s(B)$ of symmetric ASM (+1)-completions of B. We consider two cases depending on whether $k \neq l$ or k = l.

Case 1 $(k \neq l)$: Let $A' = [a'_{ij}] \in C_s(A)$. There exists l' > l such that $a'_{kl'} = +1$. We choose l' to be the smallest such integer so that $a'_{kp} = 0$ for all p with l . There also exists <math>k' > k such that $a'_{k'l} = +1$, and we choose k' to be the smallest such integer so that $a'_{ql} = 0$ for all q with k < q < k'. We then define $B' = [b'_{ij}]$ to be the matrix obtained from A' by replacing $a'_{kl}, a'_{k'l}, a'_{kl'}$, and also $a'_{lk}, a'_{lk'}, a'_{l'k}$, with 0s, and replacing $a'_{k'l'}$ (both of which must equal 0) with +1s. The matrix B' is an ASM (+1)-completion of B, and the map $f : C_s(A) \to C_s(B)$ given by $A' \to B'$ is injective. In a similar way by choosing the first +1 to the left of $a'_{kl} = -1$ we obtain another injective map $g : C_s(A) \to C_s(B)$. We have that $g(C_s(A)) \cap f(C_s(A)) = \emptyset$. Thus, in the case that $k \neq l$, each (+1)-completion of A gives two (+1)-completions of B.

Case 2 (k = l): Thus $a_{kk} = -1$ and the principal submatrix $A[k, k + 1, \ldots, n|k, k + 1, \ldots, n]$ of A determined by rows and columns $k, k + 1, \ldots, n$ has a unique -1 and this -1 is in its (1, 1)-position. Let $A' = [a'_{ij}] \in C_s(A)$. In A' there is a unique +1 to the right of $a'_{kk} = -1$, say in column r and a unique +1 below it, so in row r. The principal submatrix $A'[k, k+1, \ldots, n|k, k+1, \ldots, n]$ of A is a symmetric permutation matrix with an additional -1 in its (1, 1)-position. Let B' be the matrix obtained from A' by replacing $a'_{kk} = -1, a_{kr} = +1, a'_{rk} = +1$ with 0s and replacing $a_{rr} = 0$ with +1. Then B' is an ASM (+1)-completion of B. In a similar way, we determine in A' the largest integer p with p < k such that $a_{kp} = +1$, and thus $a_{pk} = +1$. Let B' be the matrix obtained from A' by replacing $a'_{kk} = -1$ with +1, replacing $a_{kp} = a_{kr} = a_{rk} = a_{pk} = +1$ with 0s, and replacing $a'_{rp} = a'_{pr} = 0$ with +1. Then B' is an ASM (+1)-completion of B. As before we have two injections of $C_s(A)$ into $C_s(B)$ with disjoint images, and thus each (+1)-completion of A gives two (+1)-completions of B.

Iterating the above, we see that $\pi_s(A) \leq \frac{\pi_s(C)}{2^{a+b}}$ where C is the $n \times n$ zero matrix.

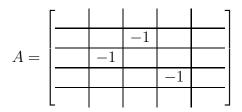
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The number of ASM (+1)-completions of C is the number of symmetric permutation matrices, and the theorem now follows.

We note that equality occurs in (1) if A = O.

In the proof of the next theorem we shall make use of an idea from [3]. Let A be an $n \times n$ (0, -1)-matrix and assume that A can be (+1)-completed to an ASM. Let $\sigma(A)$ equal the number of -1s in A. Let $Z \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ be the set of zero positions of A. The -1s of A partition Z into two families of $(n + \sigma(A))$ sets, the horizontal partition $\mathcal{H}(A) = (H_i : 1 \leq i \leq n + \sigma(A))$, consisting of the horizontal blocks, and the vertical partition $\mathcal{V}(A) = (V_i : 1 \leq i \leq n + \sigma(A))$, consisting of the *vertical blocks.* These are defined as follows: If there are $c_i \ge 0$ -1s in row *i* of *A*, then row i determines the $c_i + 1$ horizontal blocks consisting of those positions occupied by the 0s to the left of the first -1, in-between two consecutive -1s, and to the right of the last -1. The vertical blocks are defined in a similar way. Included in $\mathcal{H}(A)$ is the set of n positions in the first row and the set of n positions in the last row. Included in the vertical partition $\mathcal{V}(A)$ is the set of n positions in the first column and the set of n positions in the last column. Each $H_i \in \mathcal{H}(A)$ and each $V_i \in \mathcal{V}(A)$ intersect in at most one element of Z. The bipartite graph $G(A) \subseteq K_{n+\sigma(A),n+\sigma(A)}$ with vertex bipartition $\mathcal{H}(A), \mathcal{V}(A)$ has an edge joining $H_i \in \mathcal{H}(A)$ and $V_i \in \mathcal{V}(A)$ if and only if $H_i \cap V_j \neq \emptyset$ (and thus $|H_i \cap V_j| = 1$). As observed in [3], the matrix A has an ASM (+1)-completion if and only if the bipartite graph G(A) has a perfect matching; more specifically, if $({H_i, V_{\theta(i)}} : 1 \le i \le n + \sigma(A))$ is a perfect matching of G(A), where θ is a permutation of $\{1, 2, \ldots, n + \sigma(A)\}$, then a (+1)-completion of A to an ASM is obtained by replacing the 0s in A in the positions $\{H_i \cap V_{\theta(i)} : 1 \le i \le n + \sigma(A)\}$ with +1s.

Now suppose that A is an $n \times n$ symmetric (0, -1)-matrix. Then there is a bijection between $\mathcal{H}(A)$ and $\mathcal{V}(A)$ defined by $H_i \to V_i$ where $V_i = \{(s, r) : (r, s) \in H_i\}$ $(i = 1, 2, \ldots n + \sigma(A))$. With subscripts for the blocks in $\mathcal{H}(A)$ and $\mathcal{V}(A)$ as in this bijection, we have that $H_i \cap V_j \neq \emptyset$ if and only if $H_j \cap V_i \neq \emptyset$ $(1 \leq i, j \leq n + \sigma(A))$. Thus the $(n + \sigma(A)) \times (n + \sigma(A))$ biadjacency matrix $C = [c_{ij}]$ of the bipartite graph G(A) is symmetric and can be viewed as the adjacency matrix of a *loopy graph* $G^*(A)$ with vertex set $u_1, u_2, \ldots, u_{n+\sigma(A)}$ (u_i corresponds to both H_i and V_i) whose edges are all those pairs $\{u_i, u_j\}$ such that $H_i \cap V_j \neq \emptyset$ (equivalently, $H_j \cap V_i \neq \emptyset$). ($G^*(A)$ may have loops since it is possible that for some $i, H_i \cap V_i \neq \emptyset$ giving a loop at u_i , and thus we use the common term of loopy graph.) A *perfect matching of a loopy graph* is a collection of pairwise disjoint edges (possibly including loops) such that each vertex occurs on exactly one edge. Such a perfect matching corresponds to a symmetric permutation matrix P such that $P \leq C$ (entrywise). A perfect matching determines positions of A in which to put +1s in order to get a (+1)-completion of A to a symmetric ASM. **Example 6** Let n = 5 and consider the symmetric (0, -1)-matrix



where $\sigma(A) = 3$. Then the loopy graph $G^*(A)$ has 8 vertices and its adjacency matrix C (as usual, only the 1s are shown) is

		u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
	u_1	1	1		1		1		1
	u_2	1	1						
	u_3						1		1
C =	u_4	1							
	u_5					1	1		1
	u_6	1		1		1			
	u_7								1
	u_8	1		1		1		1	1

The loopy graph $G^*(A)$ has a perfect matching (corresponding to the shaded 1s), equivalently, there exists a symmetric permutation matrix $Q \leq C$ (entrywise), and hence there exists a (+1)-completion of A to a symmetric ASM, namely

Γ	•		+1		-]
		+1	-1	+1		
ĺ	+1	-1	+1			
		+1		-1	+1	
				+1	_	

Theorem 7 Let A be an $n \times n$ symmetric (0, -1)-matrix that has a (+1)-completion to an ASM. Then A has a (+1)-completion to a symmetric ASM.

Proof. Let the $n \times n$ matrix $B = [b_{ij}]$ be an ASM (+1)-completion of A and let

$$q(B) = \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij} - b_{ji}|.$$

Then q(B) is an even integer which counts the number of positions (i, j) with $i \neq j$ such that $b_{ij} + b_{ji} = 1$. If q(B) = 0, then B is a symmetric ASM (+1)-completion of A. Suppose that $q(B) \neq 0$. Since B is an ASM (+1)-completion of A, the +1s of B determine a perfect matching M of the bipartite graph G(A). We consider those q(B)/2 edges $\{H_i, V_j\}$ of M such that the position (p,q) of A in their intersection $H_i \cap V_j$ contains a +1 but there is not a +1 in position (q, p) of A in the intersection $H_j \cap V_i$ (so there is a +1 in the unique position in $H_j \cap V_k$ for some $k \neq j$). These q(B)/2 edges determine an asymmetric digraph D (an orientation of a graph), whose vertex set is $\{u_1, u_2, \ldots, u_{n+\sigma(A)}\}$, with no loops and at least one edge, such that any vertex with positive indegree also has positive outdegree, and vice-versa. Thus D has a directed cycle

$$\gamma: u_{i_1} \to u_{i_2} \to \dots \to u_{i_k} \to u_{i_1}$$

of length $k \geq 2$.

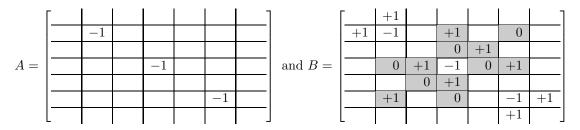
First suppose that the length k of γ is even. Then we obtain a new (+1)completion B' of A to an ASM by replacing with 0s, the +1s in positions $H_{i_1} \cap V_{i_2}, H_{i_3} \cap V_{i_4}, \ldots, H_{i_{k-1}} \cap V_{i_k}$, and by replacing with +1s, the 0s in positions $H_{i_3} \cap V_{i_2}, H_{i_5} \cap V_{i_4}, \ldots, H_{i_{k-1}} \cap V_{i_{k-2}}, H_{i_1} \cap V_{i_k}$. Moreover, q(B') < q(B).

Now suppose that k is odd. Then we claim that there is a vertex u_r of the cycle γ such that $H_r \cap V_r \neq \emptyset$, and thus $H_r \cap V_r = \{(s,s)\}$ for some s. If not, then for each i, H_i consists of positions strictly above the main diagonal or else positions strictly below the main diagonal. A similar conclusion holds for each V_i . This implies that γ has even length, a contradiction. Thus we may assume that $H_{i_1} \cap V_{i_1} = \{(s,s)\}$ and thus that the entry in B in position (s,s) is 0. Then we obtain a new (+1)-completion B' of A to an ASM by replacing the 0s in positions $H_{i_1} \cap V_{i_2}, H_{i_5} \cap V_{i_4}, \ldots, H_{i_k} \cap V_{i_{k-1}}$ with +1s and replacing the +1s in positions $H_{i_1} \cap V_{i_2}, H_{i_3} \cap H_{i_4}, \ldots, H_{i_{k-2}} \cap V_{i_{k-1}}, H_{i_k} \cap V_{i_1}$ with 0s. Again we have that q(B') < q(B). By repeating this argument, after a finite number of steps, we obtain a symmetric (+1)-completion of A to an ASM.

Another way to formulate Theorem 7 is: Let A be an $n \times n$ ASM whose -1s are in a symmetric pattern. Then there is an $n \times n$ symmetric ASM B with -1s exactly where A has -1s.

We now give two examples illustrating the argument in the proof of Theorem 7 in both the even cycle and odd cycle cases.

Example 8 Let A be the 7×7 symmetric (0, -1)-matrix and let B be the 7×7 (non-symmetric) (+1)-completion of A to an ASM as shown:



The positions which are not symmetrically occupied are shaded.

Label the sets in $\mathcal{H}(A)$ in the order of the rows and from left to right, and label the sets in $\mathcal{V}(A)$ in order of the columns and from top to bottom. Then the 10×10 biadjacency matrix C of the bipartite graph G(A) is

	1	1		1	1		1	1		1 7	
	1										
			0	1	1		1	1		1	
	1		1	1	1		1	1		1	
C =	1		1	1	0						
0 –						0	1	1		1	•
	1		1	1		1	1	1		1	
	1		1	1		1	1	0			
										1	
	1		1	1		1	1		1	1	

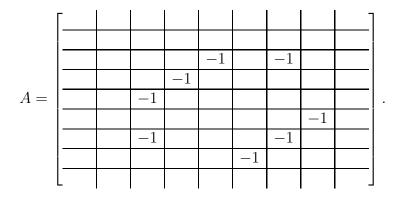
The shaded positions, where we have also shaded the corresponding diagonal elements, determine a directed cycle of even length 6 given by

$$u_3 \rightarrow u_5 \rightarrow u_4 \rightarrow u_7 \rightarrow u_6 \rightarrow u_8 \rightarrow u_3.$$

This directed cycle then determines the symmetric (+1)-completion of A to an ASM given by

		+1					
	+1	-1		+1		0	
				0	+1		
B' =		+1	0	-1	0	+1	
			+1	0			
		0		+1		-1	+1
	_					+1	

Example 9 Let A be the 9×9 symmetric (0, -1)-matrix



	_				0		+1]
					+1		0		
				+1	-1	+1	-1	+1	
			+1	-1	0		+1		
B =	0	+1	-1	+1			0		
			+1					-1	+1
	+1	0	-1	0	+1		-1	+1	
			+1			-1	+1		
	_					+1			

.

Then A has a non-symmetric ASM (+1)-completion

where the non-symmetric +1s and their symmetrically opposite 0s have been shaded. Using the same labeling procedure as in Example 8, the digraph D for this example has the directed cycle of length 3

$$u_7 \rightarrow u_{13} \rightarrow u_9 \rightarrow u_7$$

which, by using the fact that the entry in $H_9 \cap V_9 = \{(5,5)\}$ is a 0, then gives the symmetric ASM (+1)-completion of A

	-				0		+1		-
					+1		0		
				+1	-1	+1	-1	+1	
			+1	-1	0		+1		
B' =	0	+1	-1	0	+1		0		
			+1					-1	+1
	+1	0	-1	+1	0		-1	+1	
			+1			-1	+1		
	_					+1			_

We now consider an $n \times n$ (0, +1)-matrix A with at least one +1 in each row and column. In this case we consider the *horizontal partition* $\mathcal{H}^+(A) = (H_i^+ : 1 \le i \le p)$ where the H_i^+ , taken in some order, consist of those positions between two neighboring +1s in a row and, similarly, the vertical partition $\mathcal{V}^+(A) = (V_i^+ : 1 \le i \le p)$ where the V_i^+ , taken in some order, consist of those positions between two neighboring +1s in a column. As indicated, for the following reason, the number of sets p in each of the two partitions is the same: Let the row sum vector of Abe (r_1, r_2, \ldots, r_n) and let the column sum vector be $S = (s_1, s_2, \ldots, s_n)$. Then the number of sets in the horizontal partition is

$$\sum_{i=1}^{n} (r_i - 1) = \left(\sum_{i=1}^{n} r_i\right) - n = \left(\sum_{i=1}^{n} s_i\right) - n = \sum_{i=1}^{n} (s_i - 1),$$

the same as the number of sets in the vertical partition. Note that if a row (respectively, column) of A contains only one +1, then none of the positions in that row (respectively, column) are in a set of the horizontal partition (respectively, vertical partition). Let $C = [c_{ij}]$ be the $p \times p$ (0, 1)-matrix where $c_{ij} = 1$ if and only if $H_i^+ \cap V_j^+ \neq \emptyset$ ($1 \leq i, j \leq p$). The matrix C is the biadjacency matrix of a bipartite graph BG(C) with vertices bipartitioned as $\{H_i^+ : 1 \leq i \leq p\}$ and $\{V_i^+ : 1 \leq i \leq p\}$ with an edge joining H_i^+ and V_j^+ if and only if $H_i^+ \cap V_j^+ \neq \emptyset$ (and so consists of a single position). There will be a (-1)-completion of A to an ASM if and only if BG(C) has a perfect matching, equivalently, if and only if there is a permutation matrix $P \leq C$ (entrywise).

If A is symmetric, then the matrix C is a symmetric (0, 1)-matrix, possibly with 1s on the main diagonal, and so is the adjacency matrix of a a loopy graph G(C). There is a (-1)-completion of A to a symmetric ASM if and only if G(C) has a perfect matching (that is, a pairwise disjoint collection of edges and loops meeting all the vertices), that is, a symmetric permutation matrix $P \leq C$ (entrywise).

Theorem 10 Let A be an $n \times n$ symmetric (0, +1)-matrix that has a (-1)-completion to an ASM. Then A has a symmetric (-1)-completion to an ASM.

Proof. The technique of the proof is identical to the technique used in the proof of Theorem 7 and so is omitted. \Box

3 Coda

Let A be an $n \times n$ ASM. Then the row sum vector and the column sum vector of A both equal the *n*-vector $(1, 1, \ldots, 1)$ of all 1s. Let patt(A) be the (0, 1)-matrix obtained from A by replacing each entry with its absolute value. Then patt(A) is the (combinatorial) pattern of A. Because of the alternating sign property of ASMs, the pattern of an ASM uniquely determines the ASM. The pattern patt(A) of A has a row sum vector $R = (r_1, r_2, \ldots, r_n)$ and a column sum vector $S = (s_1, s_2, \ldots, s_n)$ and it is easy to verify [2] that

$$R, S \le (1, 3, 5, 7, \dots, 7, 5, 3, 1)$$
 (entrywise). (2)

Let ASM(R, S) denote the set of all ASMs whose pattern has row sum vector R and column sum vector S. In a symmetric ASM the row sum vector of its pattern equals its column sum vector. It is an open question to characterize R and S for which $ASM(R, S) \neq \emptyset$; the above conditions (2) are necessary but far from sufficient in general [2]. Let $ASM_{sym}(R)$ denote the set of all symmetric ASMs whose patterns have row sum vector, and hence column sum vector R. If an ASM A has a symmetric pattern, then A is necessarily a symmetric ASM.

The following question is motivated by a theorem of Fulkerson, Hoffman, and McAndrew (see [4] where their theorem is extended to include 1s on the main diagonal) who proved that if there is a (0, 1)-matrix with row and column sum vector R,

then there is a symmetric (0, 1)-matrix with row sum vector, and hence column sum vector, equal to R. We have been unable to answer the following ASM analogue of this theorem.

Question: Let A be an $n \times n$ ASM whose pattern has row and column sum vector equal to R. Is there a symmetric ASM whose pattern has row and column sum vector equal to R?

Let A^+ be the (0, 1)-matrix obtained from A by replacing the -1s with 0s. Then A^+ has row and column sum vector R^+ for some R^+ . Let A^- be the (0, -1)-matrix obtained from A by replacing the +1s with 0s. Then A^- has row and column sum vector R^- for some R^- . By the above theorem, there exists a symmetric (0, 1)-matrix B with row and column sum vector R^+ , and there exists a symmetric (0, -1)-matrix C with row and column sum vector R^- . We have $R^+ + R^- = (1, 1, \ldots, 1)$, but B and C need not have disjoint patterns. However, even if they did, B + C need not be an alternating sign matrix.

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